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Operators on Hilbert Space.

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ABSTRACT

Partial isometries without non-trivial reducing spaces, conjugate homogenous linear operators, and candidates for an operator without non-trivial invariant subspaces are discussed. Attention is restricted to separable Hilbert spaces.

In the first chapter a partial isometry having no non-trivial reducing space, with its null-space and that of its adjoint both of dimension one is shown to be anti-unitarily equivalent to its adjoint. The notion of an orthogonal chain of vectors with respect to a partial isometry is introduced and investigated. Examples are given based on the idea of an orthonormal basis skewed with respect to a fixed basis. The implications of an operator being anti-unitarily equivalent to its adjoint are clarified, especially in the case where the anti-unitary operator giving the equivalence is a conjugation.

In Chapter II, a structure theorem for Hermitian conjugate operators and for normal conjugate operators is given. The maximal ideal space theory is employed.

Chapter III introduces the concept of a completely normal operator and provides four candidates for an operator without a non-trivial invariant subspace, based on the polar decomposition of an operator. Pairs are given of a unitary and a Hermitian operator, both completely normal and without common non-trivial invariant subspaces.

INTRODUCTION

PREREQUISITE MATERIAL AND CONVENTIONS

Much of the contents of this section is standard. It is included for the sake of completeness.

$\mathbb{R}$ denotes the field of real numbers and $\mathbb{C}$ the field of complex numbers. If $\lambda \in \mathbb{C}$, then $\overline{\lambda}$ denotes its complex conjugate. $\mathcal{H}$ denotes a separable Hilbert space over $\mathbb{C}$. If ξ and η are vectors or elements of $\mathcal{H}$, then $(\xi|\eta)$ denotes their inner product. The norm of ξ is written $\|\xi\|$. If $\mathcal{K}$ is a Hilbert space, then we denote by dim $\mathcal{K}$ the dimension of $\mathcal{K}$, the cardinality of a complete orthonormal system of vectors in $\mathcal{K}$. In this paper the dimension of all Hilbert spaces considered is less than or equal to the cardinality of the integers. A linear manifold of $\mathcal{H}$ is a non-void subset of $\mathcal{H}$ closed with respect to vector addition and scalar multiplication. A subspace is a linear manifold closed in the topology given by the norm in $\mathcal{H}$.

If $\{\xi_a\}_{a\in A}$ is a set of vectors in $\mathcal{H}$, then $\mathcal{S}\{\xi_a\}_{a\in A}$ denotes the smallest linear manifold containing this set or the span of this set; $\mathcal{F}\{\xi_a\}_{a\in A}$ denotes the smallest subspace containing this set or its closed span. A line above a subset of $\mathcal{H}$ denotes its closure in the norm topology unless otherwise stated.

If $\mathcal{M}$ is a linear manifold, then $\mathcal{M}^{\perp}$ is the subspace of all vectors orthogonal to $\mathcal{M}$; $\mathcal{M}^{\perp} = \{\xi \in \mathcal{H} \mid (\xi|\eta) = 0 \ \forall \ \eta \in \mathcal{M}\}$. If $\{\mathcal{K}_a\}_{a\in A}$ is a pairwise orthogonal family of subspaces of $\mathcal{H}$, then $\sum\limits_{a\in A}^{\oplus} \mathcal{K}_a$ denotes the smallest subspace containing each subspace in the family. If $\mathcal{M}$ and $\mathcal{N}$ are linear manifolds, then $\mathcal{M} + \mathcal{N}$ is the smallest linear manifold containing both $\mathcal{M}$ and $\mathcal{N}$; $\mathcal{M} + \mathcal{N} = \{\xi+\eta \mid \xi \in \mathcal{M}, \eta \in \mathcal{N}\}$. We use the symbol $\perp$ to denote the orthogonality of a vector with another vector, of a vector to a linear manifold, and of one linear manifold to another. Thus, $\xi \perp \eta$ is equivalent to $(\xi|\eta) = 0$; $\xi \perp \mathcal{K}$ is equivalent to $(\xi|\eta) = 0 \ \forall \ \eta \in \mathcal{K}$; $\mathcal{L} \perp \mathcal{K}$ is equivalent to $(\xi|\eta) = 0 \ \forall \ \xi \in \mathcal{L}, \ \forall \ \eta \in \mathcal{K}$. If $\mathcal{K}$ is a linear manifold of $\mathcal{H}$, then $\mathcal{H} \ominus \mathcal{K} = \{\xi \in \mathcal{H} \mid \xi \perp \mathcal{K}\}$.

We assume familiarity with the notion of a bounded operator on $\mathcal{H}$, or more precisely, with the notion of a complex homogenous, linear, bounded operator. Since we discuss only bounded operators, we refer to them simply as operators. If A is an operator, we let $\|A\|$ denote the operator norm of A; we let A^* denote the adjoint of A, so $(A\xi|\eta) = (\xi|A^*\eta) \ \forall \ \xi,\eta \in \mathcal{H}$. If A is an operator, then we let $\mathcal{R}_A = \{A\xi \mid \xi \in \mathcal{H}\}$, the range of A. We let $\mathcal{N}_A = \{\eta \mid A\eta = 0\}$, the null-space of A. If A is an operator, then $\mathcal{R}_A^{\perp} = \mathcal{N}_{A^*}$ or $\overline{\mathcal{R}}_A = \mathcal{N}_{A^*}^{\perp}$. An operator P is a projection or a projection operator if and only if $P = P^2$ and $P = P^*$. $\mathcal{R}_P$ is a subspace; given a subspace $\mathcal{K}$ there is a projection whose range is $\mathcal{K}$. An operator H is Hermitian if and only if $H = H^*$. An operator N is normal if and only if $N^*N = NN^*$.

An operator K is an isometry or an isometric operator if and only if $\| K\xi \| = \| \xi \| \ \forall \xi \in \mathscr{H}$. An operator U is unitary if and only if it is isometric and $\mathscr{R}_U = \mathscr{H}$. If U is an operator, then U being unitary is equivalent to U*U and UU* both being the identity operator on $\mathscr{H}$. A partial isometry on $\mathscr{H}$ is an operator S such that for some subspace $\mathscr{K}$ of $\mathscr{H}$, we have $\| S\xi \| = \| \xi \| \ \forall \ \xi \in \mathscr{K}$ and $S\eta = 0$ for $\eta \perp \mathscr{K}$. $\mathscr{K}$ is denoted by $\mathscr{D}_S$. The following statements concerning an operator S are equivalent: 1) S is a partial isometry; 2) S* is a partial isometry; 3) S*S is a projection; 4) SS* is a projection. Futhermore, S*S is the projection onto $\mathscr{D}_S = \mathscr{R}_{S^*}$; SS* is the projection onto $\mathscr{D}_{S^*} = \mathscr{R}_S$.

We employ the usual concepts of invertibility, spectrum, eigenvector or proper vector, eigenvalue, approximate eigenvalue, and approximate point spectrum of an operator. We denote the spectrum of an operator A by $\sigma(A)$ and its approximate point spectrum by $\pi(A)$.

If $\mathscr{K}$ is a subset of $\mathscr{H}$ and A is an operator on $\mathscr{H}$, then $A\mathscr{K} = \{A\xi \mid \xi \in \mathscr{K}\}$. If $\mathscr{A}$ is a set of operators and $\eta_o \in \mathscr{H}$, then $\mathscr{A}\eta_o = \{A\eta_o \mid A \in \mathscr{A}\}$. If $\mathscr{A}$ is a ring of operators on a Hilbert space $\mathscr{K}$, $\xi_o \in \mathscr{K}$, and the smallest subspace containing $\mathscr{A}\xi_o$ is $\mathscr{K}$, then we say ξ_o is a cyclic vector for $\mathscr{A}$. If A is an operator on $\mathscr{K}$ and $\zeta_o \in \mathscr{K}$, then we say ζ_o is cyclic for A (for A and A*) if and only if ζ_o is cyclic for the algebraic ring generated by A (by A and A*).

A linear manifold $\mathcal{L}$ is invariant for an operator A if and only if $A\mathcal{L} \subset \mathcal{L}$; $\mathcal{L}$ reduces A if and only if $A\mathcal{L} \subset \mathcal{L}$ and $A^*\mathcal{L} \subset \mathcal{L}$. A linear manifold is invariant for a set of operators if and only if it is invariant under each operator in the set. A linear manifold reduces a set of operators if and only if it reduces each operator in the set. If a linear manifold $\mathcal{M}$ is invariant for an operator A, then the subspace $\overline{\mathcal{M}}$ is also invariant under A.

Now we turn for a moment to less common material. A conjugate homogenous, linear, bounded operator, or conjugate operator, is a mapping B of $\mathcal{K}$ into itself such that B is continuous in the norm topology and $B(\alpha\xi + \beta\eta) = \overline{\alpha}B\xi + \overline{\beta}B\eta$ $\forall\ \alpha,\beta \in \mathcal{C}$, $\forall\ \xi,\eta \in \mathcal{K}$. The norm of B is defined as for operators and is denoted $\|B\|$. If B is a conjugate operator on $\mathcal{K}$, then there is a conjugate operator B^*, the adjoint of B, satisfying $(B\xi|\eta) = (B^*\eta|\xi)$ $\forall\ \xi,\eta \in \mathcal{K}$. The existence of the adjoint of a conjugate operator is proven using the Riesz Representation theorem for bounded linear functionals on $\mathcal{K}$; the proof is analogous to that for the existence of the adjoint of an operator. See p. 38-40, [1].

If B is a conjugate operator, then $B^{**} = B$ and $(\lambda B)^* = \lambda B^*$ $\forall\ \lambda \in \mathcal{C}$. If B and C are conjugate operators, then B + C is a conjugate operator and BC is an operator. We have $(B + C)^* = B^* + C^*$. The adjoint of the operator BC is the product of the adjoints of the conjugate operators B and C in the opposite order; thus $(BC)^* = C^*B^*$. If A is an operator and B is a conjugate operator, then AB and BA are conjugate operators; $(AB)^* = B^*A^*$ and $(BA)^* = A^*B^*$. We use the symbol * above an operator or above a conjugate operator to denote its adjoint without confusion. We do not add a non-zero operator and a non-zero conjugate operator.

If B is a conjugate operator, then we define $\mathcal{R}_B$ and $\mathcal{N}_B$ as for operators. A conjugate operator H is Hermitian if and only if $H = H^*$. A conjugate operator N is normal if and only if $N^*N = NN^*$. An anti-unitary operator is a conjugate operator U such that $\| U\xi \| = \| \xi \| \ \forall \ \xi \in \mathcal{H}$ and $\mathcal{R}_U = \mathcal{H}$. A conjugate operator U is anti-unitary if and only if $U^*U = UU^* = 1$, the identity operator on $\mathcal{H}$.

We employ the terms ring and symmetric ring as in [3]. The maximal ideal space of a commutative Banach ring with identity is a compact Hausdorff space in the weakest topology in which the Gelfand transforms of ring elements are continuous; p.197, [3]. We use $\mathcal{B}(\mathcal{H})$ to denote the ring of all operators on $\mathcal{H}$. A norm-closed commutative symmetric subring of $\mathcal{B}(\mathcal{H})$ with identity is isometrically isomorphic to the ring $C(M)$ of all continuous functions on the maximal ideal space M of the subring. We use $\hat{A}(m)$ to denote the continuous function on M which is the image or Gelfand transform of an operator A in the subring. $\widehat{(A^*)}(m) = \overline{\hat{A}}(m) \ \forall \ m \in M$. $\| A \| = \sup_{m \in M} |\hat{A}(m)|$. We refer to p. 230-232, [3].

Since we are restricting our attention to separable Hilbert spaces, each maximal commutative symmetric subring of $\mathcal{B}(\mathcal{H})$ has a cyclic vector by a maximality argument. If $\mathcal{A}$ is a maximal commutative symmetric ring of operators on $\mathcal{H}$, ξ_0 is a cyclic vector for $\mathcal{A}$, and M is the maximal ideal space of $\mathcal{A}$, then $\mathcal{A}$ is isometrically isomorphic to C(M), ξ_0 corresponds to a regular Borel measure μ on M with support equal to M, and there is an isometric mapping of $\mathcal{H}$ onto $L^2(M,\mu)$. Moreover, if we denote the image of $\xi \in \mathcal{H}$ by $\xi(m) \in L^2(M,\mu)$, then we have $(A\xi)(m) = \hat{A}(m)\ \xi(m)\ \forall\ A \in \mathcal{A},\ \forall\ \xi \in \mathcal{H}$; p.247, [3].

We presuppose knowledge of the weak and strong topologies for $\mathcal{B}(\mathcal{H})$. Multiplication with one factor fixed is continuous in either of the two topologies. The transition from A to A* is continuous in the weak topology. The weak and the strong closures of symmetric subrings of $\mathcal{B}(\mathcal{H})$ coincide. A weakly closed symmetric subring of $\mathcal{B}(\mathcal{H})$ is generated by its projection operators, i.e., the minimal subring of $\mathcal{B}(\mathcal{H})$ containing all the projection operators of the given subring is the subring itself; p. 441-449, [3]. If $\mathcal{A}$ is a weakly closed symmetric commutative subring of $\mathcal{B}(\mathcal{H})$ with a cyclic vector, then $\mathcal{A}$ is a maximal commutative subring; we refer to Corollary 1.1 of [8]. A direct proof using the maximal ideal space theory without decomposition theory is possible.

We use the definitions of measure-preserving transformation, ergodic transformation, and measure algebra as in [2].

If M is the maximal ideal space of a weakly closed commutative symmetric subring of $\mathcal{B}(\mathcal{H})$, then the closure of each open set in M is open; p.31, [5]. The measure μ corresponding to a cyclic vector for a maximal such subring is a regular Borel measure. For each measurable set S, there exists a unique clopen set U for which $\mu[(S-U) \cup (U-S)] = 0$; p.48, [5]. Thus the clopen sets form a complete set of representatives for the equivalence classes which comprise the measure algebra (M,μ).

CHAPTER I

PARTIAL ISOMETRIES

Our study of partial isometries having no non-trivial reducing spaces began as an attempt to generalize the following result of von Neumann:

Let $\mathcal{H}$ be a Hilbert space of dim > 1. Let S be an isometry on $\mathcal{H}$ such that S has no non-trivial reducing space. Then $\mathcal{N}_{S*}$ is of dimension one. If $\mathcal{N}_{S*} = \mathcal{S}\{\xi_o\}$ and $\| \xi_o \| = 1$, then $\{S^i\xi_o\}_{i=0}^{\infty}$ is an orthonormal basis for $\mathcal{H}$. Consequently, $\mathcal{H}$ is of countably infinite dimension.

Outline of proof: If $\mathcal{R}_S^{\perp} = \{0\}$, then S is unitary. By the spectral theorem for normal operators and the fact that dim $\mathcal{H}$ > 1, we have that S has non-trivial reducing spaces in contradiction of the hypothesis. Hence $\mathcal{R}_S^{\perp} \neq \{0\}$.

We fix $\xi_o \in \mathcal{N}_{S*} = \mathcal{R}_S^{\perp}$ such that $\| \xi_o \| = 1$. Then for $0 \leq j < i < \infty$, we have $(S^i\xi_o | S^j\xi_o) = (S^{*j}S^i\xi_o | \xi_o) = (S^{i-j}\xi_o | \xi_o) = 0$ since $\xi_o \in \mathcal{R}_S^{\perp}$. So $(S^i\xi_o | S^j\xi_o) = \delta_{ij}$ for $0 \leq i,j < \infty$. But $\mathcal{J}\{S^i\xi_o\}_{i=0}^{\infty}$ reduces S. In fact, if $\sum_{i=0}^{\infty} a_i S^i \xi_o$ is an element of this closed span, then we have that $S^*(\sum_{i=0}^{\infty} a_i S^i \xi_o) = \sum_{i=0}^{\infty} a_{i+1} S^i \xi_o$. So $\mathcal{J}\{S^i\xi_o\}_{i=0}^{\infty} = \mathcal{H}$. |

We now give two examples of partial isometries having no non-trivial reducing spaces. Both partial isometries are defined on countably infinite dimensional Hilbert space. To facilitate the construction of these and of further examples, we will fix a method for determining an orthonormal basis for separable $\mathcal{H}$ "skewed" with respect to a given basis $\{e_i\}_{i=1}^{\infty}$.

We proceed as follows. Given the orthonormal basis $\{e_i\}_{i=1}^{\infty}$, we define a second set of vectors $\{\xi_i\}_{i=1}^{\infty}$. Let $\xi_1 = (\sqrt{2})^{-1}(e_1 + e_2)$; for $n \geq 2$, let $\xi_n = (\sqrt{2})^{-n}e_1 - \sum_{i=2}^{n}(\sqrt{2})^{i-n-2}e_i + (\sqrt{2})^{-1}e_{n+1}$. First we verify that $\| \xi_i \| = 1$, $i = 1,2,\ldots$. Clearly $\| \xi_1 \| = 1$. For $n \geq 2$, $\| \xi_n \|^2 = \| (\sqrt{2})^{-n}e_1 \|^2 + \sum_{i=2}^{n} \| (\sqrt{2})^{i-n-2}e_i \|^2 + \| (\sqrt{2})^{-1}e_{n+1} \|^2$ by the parallelogram law.

$\| \xi_n \| = 2^{-n} + \sum_{i=2}^{n} 2^{i-n-2} + 2^{-1} = 1$. So $\| \xi_n \| = 1$ for $n \geq 1$.

Now we show $(\xi_n | \xi_m) = 0$ for $1 \leq m < n < \infty$. Suppose $n \geq 2$. By definition, $\xi_n = (\sqrt{2})^{-n}e_1 - (\sqrt{2})^{-n}e_2 + \eta_n$, where $\eta_n \in \mathcal{J}\{e_i\}_{i=3}^{n+1}$. $(\eta_n | (\sqrt{2})^{-1}[e_1+e_2]) = (\eta_n | \xi_1) = 0$. Also, $(\sqrt{2})^{-n}(e_1-e_2)$ is orthogonal to $(\sqrt{2})^{-1}(e_1+e_2)$. So $\xi_1 \perp \xi_n$, $n \geq 2$. For $n > m \geq 2$, we have $(\xi_n | \xi_m) =$

$$([\sqrt{2}]^{-n}e_1 - \sum_{i=2}^{n}[\sqrt{2}]^{i-n-2}e_i + [\sqrt{2}]^{-1}e_{n+1} | [\sqrt{2}]^{-m}e_1 - \sum_{i=2}^{m}[\sqrt{2}]^{i-m-2}e_i + [\sqrt{2}]^{-1}e_{m+1}) =$$

$$[\sqrt{2}]^{-n-m} + \sum_{i=2}^{m}[\sqrt{2}]^{2i-n-m-4} - [\sqrt{2}]^{m+1-n-2-1} = 0$$

So for $n > m \geq 2$, we have $(\xi_n | \xi_m) = 0$. Hence $\{\xi_i\}_{i=1}^{\infty}$ is an orthonormal set.

It remains to show that $\mathcal{J}\{e_i\}_{i=1}^{\infty} = \mathcal{J}\{\xi_i\}_{i=1}^{\infty}$. $(e_1 | \xi_n) = [\sqrt{2}]^{-n}$, $n \geq 1$. So $\sum_{n=1}^{\infty} |(e_1 | \xi_n)|^2 = \sum_{n=1}^{\infty} \{[\sqrt{2}]^{-n}\}^2 = 1$. By Parseval's identity, we have $e_1 \in \mathcal{J}\{\xi_i\}_{i=1}^{\infty}$. Similarly, for

$m > 1$, $(e_m|\xi_n) = 0$ for $n < m-1$; $(e_m|\xi_{m-1}) = [\sqrt{2}]^{-1}$; and $(e_m|\xi_{m-1+k}) = -[\sqrt{2}]^{-k-1}$ for $k \geq 1$. Hence we have $\sum_{n=1}^{\infty} |(e_m|\xi_n)|^2 = \sum_{n=1}^{m-1} |(e_m|\xi_n)|^2 + \sum_{n=m}^{\infty} |(e_m|\xi_n)|^2 = 2^{-1} + \sum_{n=m}^{\infty} 2^{m-2-n} = 1$. So $e_m \in \mathscr{V}\{\xi_i\}_{i=1}^{\infty}$. Clearly $\mathscr{V}\{\xi_i\}_{i=1}^{\infty} \subseteq \mathscr{V}\{e_i\}_{i=1}^{\infty}$. Thus we have $\mathscr{V}\{\xi_i\}_{i=1}^{\infty} = \mathscr{V}\{e_i\}_{i=1}^{\infty}$.

Definition 1.1 If $\{e_i\}_{i=1}^{\infty}$ is an orthonormal basis for $\mathscr{H}$, then $\{\xi_i\}_{i=1}^{\infty}$, as constructed above, is also. We will refer to $\{\xi_i\}_{i=1}^{\infty}$ as the skewed basis for $\mathscr{H}$ with respect to the basis $\{e_i\}_{i=1}^{\infty}$.

While the above computations facilitate proof, they seem to obscure the idea involved in building a basis skewed with respect to a given basis. A more intuitive explanation is in order. Choose $\eta_1 = [\sqrt{2}]^{-1}[e_1+e_2]$. Then take a vector of norm one in $\mathscr{V}\{e_1, e_2\} \ominus \mathscr{V}\{\eta_1\}$, calling it η_1'. Let $\eta_2 = [\sqrt{2}]^{-1}[\eta_1' + e_3]$. Again take a vector of norm one in $\mathscr{V}\{e_1, e_2, e_3\} \ominus \mathscr{V}\{\eta_1, \eta_2\}$, calling it η_2'. Let $\eta_3 = [\sqrt{2}]^{-1}[\eta_2' + e_4]$. And so on. This process will yield a similar basis $\{\eta_i\}_{i=1}^{\infty}$ also "skewed" with respect to $\{e_i\}_{i=1}^{\infty}$. In the first explanation, we fix $\eta_1' = [\sqrt{2}]^{-1}[e_1-e_2]$; but any multiple of this vector by a complex number of modulus one would suffice.

Example 1.2 The first example is relatively simple and makes no use of a skewed basis. Consider $\mathscr{H}$ with a "double-ended" orthonormal basis $\{e_i\}_{i=-\infty}^{\infty}$. The partial isometry L_1 is defined as follows:

$$L_1(e_j) = e_{j+1} \text{ for } -\infty < j < 0 \text{ and } 1 < j < \infty$$
$$L_1(e_o) = [\sqrt{2}]^{-1}[e_1+e_2]$$
$$L_1(e_1) = 0$$

So $\mathcal{N}_{L_1} = \mathcal{S}\{e_1\}$. $\mathcal{R}^{\perp}_{L_1} = \mathcal{N}_{L_1^*} = \mathcal{S}\{[\sqrt{2}]^{-1}[e_1-e_2]\}$. Since $L_1^*L_1 = 1$ on $\mathcal{D}_{L_1}$, we have

$$L_1^*(e_{j+1}) = e_j \text{ for } -\infty < j < 0 \text{ and } 1 < j < \infty$$

We claim that L_1 has no non-trivial reducing space. We employ two steps. First we will show that e_1 is cyclic for L_1 and L_1^*. Secondly we will show that each non-zero reducing space for L_1 contains e_1. Hence L_1 will be proven to have no non-trivial reducing space.

Let $\mathcal{K}_1$ be a reducing space for L_1 containing e_1. $L_1^*(e_1) = (e_1 | [\sqrt{2}]^{-1}[e_1+e_2])e_o + (e_1 | [\sqrt{2}]^{-1}[e_1-e_2])0 = [\sqrt{2}]^{-1}e_o$. $L_1^{*n}(e_1) = [\sqrt{2}]^{-1}e_{-n+1}$ for $n \geq 1$. So $\mathcal{S}\{e_i\}_{i=-\infty}^{1} \subset \mathcal{K}_1$.

$L_1(e_o) = [\sqrt{2}]^{-1}[e_1+e_2]$. $L_1([\sqrt{2}]^{-1}[e_1+e_2]) = ([\sqrt{2}]^{-1}[e_1+e_2] | e_1)0 + ([\sqrt{2}]^{-1}[e_1+e_2] | e_2)e_3 = [\sqrt{2}]^{-1}e_3$. $L_1^n([\sqrt{2}]^{-1}[e_1+e_2] = [\sqrt{2}]^{-1}e_{n+2}$ for $n \geq 1$. Lastly. $L_1^*L_1([\sqrt{2}]^{-1}[e_1+e_2] = [\sqrt{2}]^{-1}e_2$, since $L_1^*L_1$ is the projection onto $\mathcal{D}_{L_1}$. So $\mathcal{S}\{e_i\}_{i=2}^{\infty} \subset \mathcal{K}_1$. Hence $\mathcal{K} = \mathcal{K}_1$, and the first step is completed.

Now let $\mathcal{K}_1'$ be a non-zero reducing space for L_1. We must show $e_1 \in \mathcal{K}_1'$. Pick $\zeta \in \mathcal{K}_1'$, $\zeta \neq 0$. Let $\zeta = \sum_{-\infty}^{\infty} a_i e_i$. $\zeta \neq 0$ implies $a_m \neq 0$ for some m; we will assume $m \geq 1$. The case where $m < 1$ is treated in a similar fashion. If $m = 1$, then $\zeta - L_1^*L_1\zeta = (1 - L_1^*L_1)\,\zeta = \sum_{-\infty}^{\infty} a_i e_i - \sum_{\substack{-\infty \\ i \neq 1}}^{\infty} a_i e_i = -a_1 e_1$, If $m > 1$, let p be the least integer of those $m > 1$ such that $a_m \neq 0$. Thus $\zeta = \sum_{-\infty=i}^{0} a_i e_i + \sum_{i=p}^{\infty} a_i e_i$. $L_1^{*p-2}(\zeta) = \sum_{-\infty}^{2-p} a_{i+p-2} e_i + \sum_{i=2}^{\infty} a_{i+p-2} e_i$. Now $a_{2+p-2} = a_p$. So $(1-L_1 L_1^*) L_1^{*p-2}\zeta = (L_1^{*p-2}\zeta \mid [\sqrt{2}]^{-1}[e_1-e_2])[\sqrt{2}]^{-1}[e_1-e_2] = -[\sqrt{2}]\, a_p[e_1-e_2]$. So $[\sqrt{2}]^{-1}[e_1-e_2] \in \mathcal{K}_1'$. But $(1-L_1^* L_1)[\sqrt{2}]^{-1}[e_1-e_2] = [\sqrt{2}]^{-1} e_1$. We then have $e_1 \in \mathcal{K}_1'$ and $\mathcal{K}_1' = \mathcal{H}$.

We have shown that L_1 is a partial isometry with no non-trivial reducing space, $\dim \mathcal{N}_{L_1} = \dim \mathcal{N}_{L_1^*} = 1$. Also, we note that $\mathcal{N}_{L_1} = \mathcal{S}\{e_1\}$ is not cyclic for L_1^*. In fact, $e_2 \perp L_1^{*n} e_1$ for $n \geq 0$. Similarly, $\mathcal{N}_{L_1^*}$ is not cyclic for L_1.

Example 1.3 Our second example makes use of a skewed basis. Let $\mathcal{H}$ be a Hilbert space with orthonormal basis $\{e_i\}_{i=0}^{\infty}$. With respect to the orthonormal set $\{e_i\}_{i=1}^{\infty}$, let $\{\xi_i\}_{i=1}^{\infty}$ be the skewed basis. We recall that $\xi_1 = [\sqrt{2}]^{-1}[e_1+e_2]$, $\xi_n = [\sqrt{2}]^{-n} e_1 - \sum_{i=2}^{n} [\sqrt{2}]^{i-n-2} e_i + [\sqrt{2}]^{-1} e_{n+1}$ for $n \geq 2$, and $\mathcal{J}\{\xi_i\}_{i=1}^{\infty} = \mathcal{J}\{e_i\}_{i=1}^{\infty}$. We define L_2 as follows:

$$L_2(e_o) = \xi_1$$
$$L_2(e_i) = \xi_i \text{ for } i \geq 2$$

So $\mathcal{N}_{L_2} = \mathcal{S}\{e_1\}$ and $\mathcal{R}_{L_2} = \mathcal{J}\{\xi_i\}_{i=1}^{\infty}$. Hence $\mathcal{R}_{L_2}^{\perp} = \mathcal{N}_{L_2^*} = \mathcal{S}\{e_o\}$. To show that L_2 has no non-trivial reducing space, we proceed in two steps. First, we show that e_o is cyclic for L_2. Second, we show that a non-zero reducing space for L_2 contains e_o.

Let $\mathcal{K}_2$ be a subspace invariant under L_2 and containing e_o. If $\xi_i \in \mathcal{K}_2$ for $1 \leq i < \infty$, then we have that $\mathcal{K}_2 = \mathcal{H}$ since

$\mathscr{S}\{e_o\} \oplus \mathscr{S}\{\xi_i\}_{i=1}^{\infty} = \mathscr{K}$. Obviously $\xi_1 = L_2(e_o) \in \mathscr{K}_2$. Also, $L_2(\xi_1) = L_2([\sqrt{2}]^{-1}[e_1+e_2]) = L_2([\sqrt{2}]^{-1}e_2) = [\sqrt{2}]^{-1}\xi_2$. Now suppose for some $j \geq 2$ we have $\xi_j \notin \mathscr{K}_2$. Let q be the least integer of those j for which $\xi_j \notin \mathscr{K}_2$. Thus $\mathscr{S}\{\xi_i\}_{i=1}^{q-1} \subset \mathscr{K}_2$ but $\xi_q \notin \mathscr{K}_2$. Since $q \geq 3$, we have $q-1 \geq 2$ and

$$\xi_{q-1} = [\sqrt{2}]^{1-q}e_1 - \sum_{i=2}^{q-1}[\sqrt{2}]^{i-q-1}e_i + [\sqrt{2}]^{-1}e_q$$
$$L_2(\xi_{q-1}) = L_2([\sqrt{2}]^{1-q}e_1) - \sum_{i=2}^{q-1}[\sqrt{2}]^{i-q-1}L_2(e_i) + [\sqrt{2}]^{-1}L_2(e_q) =$$
$$0 - \sum_{i=2}^{q-1}[\sqrt{2}]^{i-q-1}\xi_i + [\sqrt{2}]^{-1}\xi_q$$

Now $L_2(\xi_{q-1}) \in \mathscr{K}_2$. But $\{\xi_i\}_{i=1}^{q-1} \subset \mathscr{K}_2$ by choice of q. So $\xi_q \in \mathscr{K}_2$ and we reach a contradiction. Hence $\mathscr{S}\{\xi_i\}_{i=1}^{\infty} \subset \mathscr{K}_2$ and $\mathscr{K}_2 = \mathscr{K}$.

Secondly, let $\mathscr{K}_2'$ be a non-zero reducing subspace for L_2. We must show $e_o \in \mathscr{K}_2'$. Let $\zeta \in \mathscr{K}_2'$, $\zeta \neq 0$, $\zeta = a_o e_o + \sum_{i=1}^{\infty} a_i \xi_i$. If $a_o \neq 0$, then $(1 - L_2L_2^*)\zeta = a_o e_o \neq 0$. Otherwise, we note that $(L_2^*\xi_i|\xi_{i-1}) = (e_i|\xi_{i-1}) = [\sqrt{2}]^{-1} \neq 0$ for $i \geq 2$. Also, $(L_2^*\xi_i|\xi_k) = 0 = (e_i|\xi_k)$ for $1 \leq k < i-1$. Hence if $\zeta = \sum_{i=s}^{\infty} a_i\xi_i$ with $a_s \neq 0$, we have $(L_2^{*s-1}\zeta|[\sqrt{2}]^{-1}[e_1+e_2]) = [\sqrt{2}]^{-s}a_s$ and $(L_q^{*s}\zeta|e_o) = [\sqrt{2}]^{-s}a_s$. Thus $(1-L_2L_2^*)L_2^{*s}\zeta = [\sqrt{2}]^{-s}a_s e_o$. In any case, we have shown that $e_o \in \mathscr{K}_2'$. So $\mathscr{K}_2' = \mathscr{K}$, and we see that L_2 has no non-trivial reducing space. Also, dim $\mathscr{N}_{L_2} = 1 =$ dim $\mathscr{N}_{L_2^*}$; and e_o is cyclic for L_2. As a result of lemma 1.41 to be proven later, we know that $\mathscr{N}_{L_2^*}$ being cyclic for L_2 implies $\mathscr{N}_{L_2}$ is cyclic for L_2^*.

<u>Theorem 1.4</u> Let U be a partial isometry such that dim $\mathscr{N}_U =$ dim $\mathscr{N}_{U^*} = 1$. Let $\mathscr{N}_U = \mathscr{S}\{e_o\}$; let $\mathscr{N}_{U^*} = \mathscr{S}\{\eta_o\}$. Then for each

pair $P(x)$, $Q(x)$ of polynomials over the complex numbers, we have $(P(U^*)e_o|Q(U^*)e_o) = \overline{(\overline{P}(U)\eta_o|\overline{Q}(U)\eta_o)} = (\overline{Q}(U)\eta_o|\overline{P}(U)\eta_o)$ where $\overline{P}(x)$ denotes the polynomial obtained from $P(x)$ by conjugation of coefficients.

Proof: We recall that UU^*, U^*U, $1-UU^*$, $1-U^*U$ are the projections onto $\mathcal{R}_{U^*}$, $\mathcal{R}_U$, $\mathcal{N}_{U^*}$, and $\mathcal{N}_U$, respectively. $(e_o|e_o) = 1 = \overline{(\eta_o|\eta_o)}$. So the assertion is true for constant polynomials. We suppose that for $0 \leq i,j \leq n-1$ it is true that $(U^{*i}e_o|U^{*j}e_o) = \overline{(U^i\eta_o|U^j\eta_o)}$. We will show that this equality holds for $0 \leq i,j \leq n$. If $i = j = 0$, then the equality is clear. If $i = 0$ and $j \neq 0$ or if $i \neq 0$ and $j = 0$, then both sides of the equality are zero, since $\mathcal{R}_{U^*}\perp\mathcal{N}_U$ and $\mathcal{R}_U\perp\mathcal{N}_{U^*}$. Thus we can suppose that $0 < i,j \leq n$. $(U^{*i}e_o|U^{*j}e_o) = (UU^*U^{*i-1}e_o|U^{*j-1}e_o)$ $= (U^{*i-1}e_o - U^{*i-1}e_o + UU^*U^{*i-1}e_o|U^{*j-1}e_o) = (U^{*i-1}e_o|U^{*j-1}e_o)$ $- ([1-UU^*]U^{*i-1}e_o|U^{*j-1}e_o) = (U^{*i-1}e_o|U^{*j-1}e_o) -$ $(\,(U^{*i-1}e_o|\eta_o)\eta_o|U^{*j-1}e_o) = (U^{*i-1}e_o|U^{*j-1}e_o) -$ $(U^{*i-1}e_o|\eta_o)\cdot(\eta_o|U^{*j-1}e_o)$.

By symmetry in the above computations, it is clear that $(U^i\eta_o|U^j\eta_o) = (U^{i-1}\eta_o|U^{j-1}\eta_o) - (U^{i-1}\eta_o|e_o)\cdot(e_o|U^{j-1}\eta_o)$. However, by our induction hypothesis $\overline{(U^{i-1}\eta_o|U^{j-1}\eta_o)} = (U^{*i-1}e_o|U^{*j-1}e_o)$. Clearly $\overline{(U^{i-1}\eta_o|e_o)}\cdot\overline{(e_o|U^{j-1}\eta_o)} = \overline{(\eta_o|U^{*i-1}e_o)}\cdot\overline{(U^{*j-1}e_o|\eta_o)} = (U^{*i-1}e_o|\eta_o)\cdot(\eta_o|U^{*j-1}e_o)$. So $(U^{*i}e_o|U^{*j}e_o) = \overline{(U^i\eta_o|U^j\eta_o)}$ for $0 \leq i,j < \infty$ by induction. Now let $P(x) = \sum_{i=0}^{n} a_i x^i$; let $Q(x) = \sum_{j=0}^{m} b_j x^j$. $(P(U^*)e_o|Q(U^*)e_o)$ $= \sum_{i,j} a_i\overline{b}_j(U^{*i}e_o|U^{*j}e_o) = \sum_{i,j} a_i\overline{b}_j\overline{(U^i\eta_o|U^j\eta_o)} =$

$\sum_{i,j} a_i \overline{b}_j (U^j \eta_o | U^i \eta_o) = \sum_{i,j} (\overline{b}_j U^j \eta_o | \overline{a}_i U^i \eta_o) = (\overline{Q}(U)\eta_o | \overline{P}(U)\eta_o) = \overline{(\overline{P}(U)\eta_o | \overline{Q}(U)\eta_o)}$. |

Lemma 1.41 Let U be a partial isometry on $\mathcal{H}$ such that U has no non-trivial reducing space and dim $\mathcal{N}_U$ = dim $\mathcal{N}_{U^*}$ = 1. Let $\mathcal{N}_U = \mathcal{S}\{e_o\}$; let $\mathcal{N}_{U^*} = \mathcal{S}\{\eta_o\}$. Then $\mathcal{J}\{U^i \eta_o\}_{i=0}^{\infty} = \mathcal{H}$ is equivalent to $\mathcal{J}\{U^{*i} e_o\}_{i=0}^{\infty} = \mathcal{H}$.

Proof: By the symmetry in the following argument, it will be clear that it is enough to show that $\mathcal{J}\{U^i \eta_o\}_{i=0}^{\infty} = \mathcal{H}$ implies that $\mathcal{J}\{U^{*i} e_o\}_{i=0}^{\infty} = \mathcal{H}$. We assume that $\mathcal{J}\{U^i \eta_o\}_{i=0}^{\infty} = \mathcal{H}$. Thus there exists a sequence of polynomials over the complex numbers, $\{P_i(x)\}_{i=1}^{\infty}$, such that $\| P_i(U)\eta_o - e_o \|^2 < 2^{-i}$. By theorem 1.4 we know that $\| P_i(U)\eta_o \|^2 = \| \overline{P}_i(U^*)e_o \|^2$. So $\| P_i(U)\eta_o - e_o \|^2 = \| P_i(U)\eta_o \|^2 + \| e_o \|^2 - (P_i(U)\eta_o | e_o) - (e_o | P_i(U)\eta_o) = \| \overline{P}_i(U^*)e_o \|^2 + \| \eta_o \|^2 - (\eta_o | \overline{P}_i(U^*)e_o) - (\overline{P}_i(U^*)e_o | \eta_o) = \| \overline{P}_i(U^*)e_o - \eta_o \|^2 \leq 2^{-i}$. Hence $\eta_o \in \mathcal{J}\{U^{*i} e_o\}_{i=0}^{\infty}$. Let $\mathcal{K} = \mathcal{J}\{U^{*i} e_o\}_{i=0}^{\infty}$. Clearly $\mathcal{K}$ is invariant under U*. Since $\mathcal{R}_{U^*} \perp \mathcal{N}_U$, we can write $\mathcal{K} = \mathcal{S}\{e_o\} \oplus U^*\mathcal{K}$. But $\eta_o \in \mathcal{K}$ implies U* is a partial isometry when U* is restricted to $\mathcal{K}$. So the domain of $U^*|\mathcal{K}$ is $\mathcal{K} \ominus \mathcal{S}\{\eta_o\} = \mathcal{K} \cap \mathcal{D}_{U^*}$. So we have $\mathcal{K} = \mathcal{S}\{e_o\} \oplus U^*(\mathcal{K} \cap \mathcal{D}_{U^*})$. If $\xi \in \mathcal{K}$, then $\xi = ae_o + U^*\xi_1$ where $a \in \mathbb{C}$ and $\xi_1 \in \mathcal{K} \cap \mathcal{D}_{U^*}$. So $U\xi = U(ae_o) + UU^*\xi_1 = 0 + UU^*\xi_1 = \xi_1$. So $\mathcal{K}$ is invariant under U as well as under U*. Therefore $\mathcal{K} = \mathcal{H}$. As noted in the beginning of the proof, we can interchange the roles of U and U* along with those of e_o and η_o in the above argument. Thus the lemma is proven. |

We now introduce a definition of a chain of vectors with respect to a partial isometry U. Von Neumann's theorem on isometries without non-trivial reducing spaces means that each such isometry generates a chain whose span is dense in the Hilbert space.

Definition 1.5 A chain with respect to a partial isometry U is a sequence of vectors of the form $\{U^i\xi_0\}_{i=0}^{\infty}$ with the requirement that $(U^i\xi_0|U^j\xi_0) = \delta_{ij}$, $0 \leq i,j < \infty$.

Theorem 1.6 Let U be a partial isometry on $\mathcal{H}$ such that U has no non-trivial reducing space. Then the following statements are equivalent:

(a) $\mathcal{J}\{U^{*i}\eta_U\}_{i=0}^{\infty} \neq \mathcal{H}$;

(b) There exists $\xi_0 \in \mathcal{H}$ such that $\| U^i\xi_0 \| = 1$ for $0 \leq i < \infty$;

(c) There is a chain with respect to U, a U-chain.

Proof:

(a) implies (b). Since $\mathcal{H} \ominus \mathcal{J}\{U^{*i}\eta_U\}_{i=0}^{\infty} \neq \{0\}$, we can pick ξ_0 in this subspace such that $\| \xi_0 \| = 1$. Now $\mathcal{H} \ominus \mathcal{J}\{U^{*i}\eta_U\}_{i=0}^{\infty}$ is invariant with respect to U and is orthogonal to η_U. So U restricted to this subspace is an isometry. Hence (b) follows.

(b) implies (c). Let $\mathcal{K} = \mathcal{J}\{U^i\xi_0\}_{i=0}^{\infty}$. $\mathcal{K}$ is invariant with respect to U. Also, U is an isometry when restricted to $\mathcal{K}$. Let $\mathcal{L} = \mathcal{K} \ominus U\mathcal{K}$. We suppose $\mathcal{L} = \{0\}$. Then $\mathcal{K} = U\mathcal{K}$ or $U^*\mathcal{K} = U^*U\mathcal{K} = \mathcal{K}$ since $\mathcal{K} \perp \eta_U$ and $U^*U = 1$ on $\mathcal{D}_U$. $\mathcal{K}$ is thus a reducing space, contradicting the hypothesis about U. So $\mathcal{L} \neq \{0\}$. Let $\xi_1 \in \mathcal{L}$ such that $\| \xi_1 \| = 1$. For $i > j$, $(U^i\xi_1|U^j\xi_1) =$

$(U^{*j}U^i\xi_1|\xi_1) = (U^{i-j}\xi_1|\xi_1) = 0$ since $\xi_1 \in \mathcal{K}\ominus U\mathcal{K}$. And $\mathcal{N}_U \perp \mathcal{K}$ implies that $(U^i\xi_1|U^i\xi_1) = 1$ for $0 \leq i < \infty$. Thus $(U^i\xi_1|U^j\xi_1) = \delta_{ij}$ for $0 \leq i,j < \infty$.

(c) implies (a). Let ξ_1 be such that $(U^i\xi_1|U^j\xi_1) = \delta_{ij}$ for $0 \leq i,j < \infty$. So $U^i\xi_1 \perp \mathcal{N}_U$ for $0 \leq i < \infty$. So $\mathcal{N}_U \perp \mathcal{J}\{U^i\xi_1\}_{i=0}^{\infty}$. This closed span is invariant with respect to U. So we have $\mathcal{H}\ominus \mathcal{J}\{U^i\xi_1\}_{i=0}^{\infty}$ is invariant with respect to U* and contains $\mathcal{N}_U$. Thus $\mathcal{J}\{U^{*i}\mathcal{N}_U\}_{i=0}^{\infty} \subset \mathcal{H}\ominus \mathcal{J}\{U^i\xi_1\}_{i=0}^{\infty} \neq \mathcal{H}$. |

<u>Lemma 1.61</u> Suppose U is a partial isometry on $\mathcal{H}$, $\mathcal{K}$ is a subspace of $\mathcal{H}$, and $U\mathcal{K} \subset \mathcal{K}$. Then $\mathcal{K}$ is a reducing subspace for U if and only if $\mathcal{K}\ominus U\mathcal{K} \subset \mathcal{N}_{U*}$ and $\mathcal{N}_U = (\mathcal{N}_U \cap \mathcal{K})\oplus(\mathcal{N}_U \ominus \mathcal{K}) = (\mathcal{N}_U \cap \mathcal{K})\oplus(\mathcal{N}_U \cap \mathcal{K}^{\perp})$.

Proof: We suppose $\mathcal{K}$ reduces U. Then $\mathcal{K}$ and $\mathcal{K}^{\perp}$ are invariant with respect to $1 - U^*U$, the projection onto $\mathcal{N}_U$. Thus $\mathcal{N}_U = (1-U^*U)\mathcal{H} = (1-U^*U)(\mathcal{K}\oplus\mathcal{K}^{\perp}) = [(1-U^*U)\mathcal{K}]\oplus[(1-U^*U)\mathcal{K}^{\perp}] = (\mathcal{N}_U \cap \mathcal{K})\oplus(\mathcal{N}_U \ominus \mathcal{K})$. Similarly, since $U\mathcal{K} \subset \mathcal{K}$ and $U\mathcal{K}^{\perp} \subset \mathcal{K}^{\perp}$, we have that $\mathcal{K}\ominus U\mathcal{K} \subset (\mathcal{K}\ominus U\mathcal{K})\oplus(\mathcal{K}^{\perp}\ominus U\mathcal{K}^{\perp}) = \mathcal{H}\ominus(U\mathcal{K}\oplus U\mathcal{K}^{\perp}) = \mathcal{H}\ominus U\mathcal{H} = \mathcal{N}_{U*}$.

Now we suppose that $\mathcal{K}$ is an invariant subspace for U such that $\mathcal{K}\ominus U\mathcal{K} \subset \mathcal{N}_{U*}$ and $\mathcal{N}_U = (\mathcal{N}_U \cap \mathcal{K})\oplus(\mathcal{N}_U \ominus \mathcal{K})$. We must show $U^*\mathcal{K} \subset \mathcal{K}$. $\mathcal{K}\ominus \mathcal{N}_U = \mathcal{K}\ominus[(\mathcal{N}_U \cap \mathcal{K})\oplus(\mathcal{N}_U \ominus \mathcal{K})] = \mathcal{K}\ominus(\mathcal{N}_U \cap \mathcal{K})$ since $\mathcal{K}\ominus(\mathcal{N}_U \ominus \mathcal{K}) = \{0\}$. Thus $\mathcal{K}\cap \mathcal{D}_U = \mathcal{K}\ominus(\mathcal{N}_U \cap \mathcal{K})$ or $\mathcal{K} = (\mathcal{K}\cap \mathcal{D}_U)\oplus(\mathcal{K}\cap \mathcal{N}_U)$. Now $U\mathcal{K} = U(\mathcal{K}\cap \mathcal{D}_U)$. $\mathcal{K}\ominus U\mathcal{K} \subset \mathcal{N}_{U*}$ implies that $\mathcal{K} = U\mathcal{K}\oplus(\mathcal{K}\ominus U\mathcal{K}) = U\mathcal{K}\oplus(\mathcal{K}\cap \mathcal{N}_{U*}) = U(\mathcal{K}\cap \mathcal{D}_U)\oplus(\mathcal{K}\cap \mathcal{N}_{U*})$. Thus $U^*\mathcal{K} = U^*[U(\mathcal{K}\cap \mathcal{D}_U)\oplus(\mathcal{K}\cap \mathcal{N}_{U*})] = U^*U(\mathcal{K}\cap \mathcal{D}_U) + U^*(\mathcal{K}\cap \mathcal{N}_{U*}) =$

$U^*U(\mathcal{K} \cap \mathcal{D}_U) = \mathcal{K} \cap \mathcal{D}_U \subset \mathcal{K}$. So $U^*\mathcal{K} \subset \mathcal{K}$ and $\mathcal{K}$ reduces U. |

Theorem 1.7 Let U be a partial isometry on $\mathcal{H}$ such that U has no non-trivial reducing space and $\dim \mathcal{N}_U = \dim \mathcal{N}_{U^*} = 1$. Let $\mathcal{N}_U = \mathcal{S}\{e_o\}$ and $\mathcal{N}_{U^*} = \mathcal{S}\{\eta_o\}$. Then there exists a unique anti-unitary operator S on $\mathcal{H}$ satisfying the conditions $SUS^{-1} = U^*$ and $S\eta_o = e_o$. Also we have $S^2\eta_o = \eta_o$. [We will see later that under these conditions $S^2\eta_o = \eta_o$ implies that $S^2 = 1$.]

Proof: Since $S^{-1} = S^*$, we see that $SUS^* = U^*$ is equivalent to $S^{**}U^*S^* = U$. This is in turn equivalent to $S^*US = U^*$ or $S^{-1}US = U^*$. Let $\mathcal{J}\{U^i\eta_o\}_{i=0}^{\infty} = \mathcal{L}$. We consider two cases: $\mathcal{L} = \mathcal{H}$ and $\mathcal{L} \neq \mathcal{H}$.

First we suppose that $\mathcal{L} = \mathcal{H}$. By lemma 1.41 $\mathcal{J}\{U^{*i}e_o\}_{i=0}^{\infty} = \mathcal{H}$. We define $S\eta_o = e_o$. Thus $\eta_o = S^{-1}e_o$. Now if $SUS^{-1} = U^*$, then $SU^nS^{-1} = U^{*n}$ for $n \geq 0$. If S is conjugate homogenous and linear and if P(x) is a polynomial over $\mathcal{C}$, then we have $SP(U)S^{-1} = \overline{P}(U^*)$. Thus if it is possible to define S meeting the requirements of the theorem, we see that $SP(U)S^{-1}e_o = \overline{P}(U^*)e_o$ or $S[P(U)\eta_o] = \overline{P}(U^*)e_o$ on $\mathcal{S}\{U^i\eta_o\}_{i=0}^{\infty}$. By theorem 1.4, for $\xi_1, \xi_2 \in \mathcal{L}$, we have $\| S\xi_1 \| = \| \xi_1 \|$; so S is well-defined. By the same theorem $(S\xi_1|S\xi_2) = \overline{(\xi_1|\xi_2)}$. S is conjugate homogenous and linear. We extend the domain of definition of S to $\mathcal{J}\{U^i\eta_o\}_{i=0}^{\infty} = \mathcal{L} = \mathcal{H}$. On this closed span we clearly have $SUS^{-1} = U^*$. SUS^{-1} is a bounded linear operator which equals U^* on a dense linear manifold of $\mathcal{H}$ and hence equals U^* on the entire space. The range of S includes $\mathcal{S}\{U^{*i}e_o\}_{i=0}^{\infty}$ and hence is $\mathcal{H}$. Therefore, S is anti-unitary; the theorem is proven in the first case.

Now we suppose $\mathcal{L} \neq \mathcal{H}$. By lemma 1.41 $\mathcal{J}\{U^{*i}e_o\}_{i=0}^{\infty} \neq \mathcal{H}$. Let $\mathcal{M} = \mathcal{J}\{U^{*i}e_o\}_{i=0}^{\infty}$. If $e_o \in \mathcal{L}$, then lemma 1.61 implies that $\mathcal{L}$ reduces U. So $e_o \notin \mathcal{L}$. Let P and Q be the projections onto $\mathcal{L}$ and $\mathcal{M}$ respectively. Let $\mathcal{L}' = \mathcal{L} \oplus \mathcal{S}\{e_o\}$. Let $\xi_o = [\|e_o-Pe_o\|]^{-1}[e_o-Pe_o]$. $\mathcal{L}' \ominus \mathcal{L} = \mathcal{S}\{\xi_o\}$. $\mathcal{L} = \mathcal{S}\{\eta_o\} \oplus U\mathcal{L}$, since $\mathcal{R}_U \perp \mathcal{N}_{U^*}$. $U\mathcal{L}' = U\mathcal{L}$. $\mathcal{L}' = \mathcal{L} \oplus \mathcal{S}\{\xi_o\} = \mathcal{S}\{\eta_o\} \oplus U\mathcal{L}' \oplus \mathcal{S}\{\xi_o\}$. Since $\mathcal{N}_U \subset \mathcal{L}'$, we can write $\mathcal{L}' = \mathcal{S}\{\eta_o\} \oplus U\{\mathcal{L}' \cap \mathcal{D}_U\} \oplus \mathcal{S}\{\xi_o\}$. With these facts we will now show that $\mathcal{H} = \mathcal{L} \oplus \mathcal{J}\{U^{*i}\xi_o\}_{i=0}^{\infty}$. $U\xi_o \in U\mathcal{L}' = U\mathcal{L} \subset \mathcal{L}$. If $\xi \in \mathcal{L}'$ and $\xi \perp \xi_o$, then we have $\xi = a\eta_o + U\xi_2$, where $a \in \mathcal{C}$ and $\xi_2 \in \mathcal{L}' \cap \mathcal{D}_U$. Hence $U^*\xi = aU^*\eta_o + U^*U\xi_2 = \xi_2 \in \mathcal{L}'$. So $U^*\mathcal{L} \subset \mathcal{L}'$. $\xi_o \perp \mathcal{L}$ implies $\xi_o \perp U^n\mathcal{L}$ for $n \geq 1$. $U^n\mathcal{L} = U^n\mathcal{L}'$. So $U^{*n}\xi_o \perp \mathcal{L}'$ for $n \geq 1$. Since $\mathcal{N}_{U^*} \subset \mathcal{L}$, we see that $(U^*\xi_o|\xi_o) = 0$ and the norm of $U^*\xi_o$ is equal to one. More generally, $(U^{*i}\xi_o|U^{*j}\xi_o) = \delta_{ij}$ for $0 \leq i,j \leq \infty$. Let $\zeta \in \mathcal{L} \oplus \mathcal{J}\{U^{*i}\xi_o\}_{i=0}^{\infty}$, $\zeta = \xi_1 + \sum_{i=0}^{\infty} a_i U^{*i}\xi_o$ where $\xi_1 \in \mathcal{L}$. $U\zeta = U\xi_1 + a_oU\xi_o + \sum_{i=0}^{\infty} a_{i+1}U^{*i}\xi_o$. But we noted earlier $\xi_1 \in \mathcal{L}$ implies $U\xi_1 \in \mathcal{L}$ and $U\xi_o \in U\mathcal{L}' \subset \mathcal{L}$. Thus $U\zeta \in \mathcal{L} \oplus \mathcal{J}\{U^{*i}\xi_o\}_{i=0}^{\infty}$. $U^*\zeta = U^*\xi_1 + \sum_{i=1}^{\infty} a_{i-1}U^{*i}\xi_o$. We noted above that $\xi_1 \in \mathcal{L}$ implies that $U^*\xi_1 \in \mathcal{L} \oplus \mathcal{S}\{\xi_o\}$. Thus $U^*\zeta \in \mathcal{L} \oplus \mathcal{J}\{U^{*i}\xi_o\}_{i=0}^{\infty}$, and so this space reduces U. Hence $\mathcal{L} \oplus \mathcal{J}\{U^{*i}\xi_o\}_{i=0}^{\infty} = \mathcal{H}$.

Similarily, if we set $\zeta_o = \|\eta_o - Q\eta_o\|^{-1}[\eta_o - Q\eta_o]$, then we have $(U^i\zeta_o|U^j\zeta_o) = \delta_{ij}$ for $0 \leq i,j < \infty$ and likewise

$\mathcal{M} \oplus \mathcal{J}\{U^i\zeta_o\}_{i=0}^{\infty} = \mathcal{K}$. By the proof of lemma 1.41, for a polynomial $R(x)$ we have $\|R(U)\eta_o - e_o\|^2 = \|\overline{R}(U^*)e_o - \eta_o\|^2$. So if $\{R_i(x)\}_{i=1}^{\infty}$ is a sequence of polynomials such that $R_i(U)\eta_o$ converges (norm) to Pe_o as i goes to infinity, then $R_i(U^*)e_o$ converges to $Q\eta_o$. Here we use the elementary fact that the projection of a vector η_1 onto a subspace $\mathcal{J}$ is that vector η_2 of $\mathcal{J}$ such that $\inf_{\eta\in\mathcal{J}}\|\eta-\eta_1\|$ is attained. Thus $S(Pe_o) = Q\eta_o$, where we define S as in the first case. In this case, S is a conjugate homogenous linear isometry mapping $\mathcal{L}$ onto $\mathcal{M}$; S must be extended to meet the requirements of the theorem. For $\xi \in \mathcal{M}$ we have $SUS^{-1}\xi = U^*\xi$, just as in case one.

Now we enlarge the domain of definition of S. As we shall see, this extension depends only on our definition of $S(\xi_o)$ or of $S(e_o - Pe_o)$. First we show that $S(e_o - Pe_o)$ can be defined in at most one way to satisfy the theorem. From the equation $SUS^{-1} = U^*$ we see that $S^{-1}\mathcal{N}_{U^*} = \mathcal{N}_U$ or $\mathcal{N}_{U^*} = S\mathcal{N}_U$. Since S is norm-preserving or isometric, we see that $Se_o = a\eta_o$, where $|a| = 1$, $a \in \mathcal{C}$. Considering that $SPe_o = Q\eta_o$, we obtain that $0 = (e_o-Pe_o|Pe_o) = (Se_o-SPe_o|SPe_o) = (a\eta_o-Q\eta_o|Q\eta_o) = a(\eta_o|Q\eta_o) - (Q\eta_o|Q\eta_o) = [a - 1](Q\eta_o|Q\eta_o)$. So $a = 1$ if $(Q\eta_o|Q\eta_o) \neq 0$; but if $\eta_o \perp \mathcal{M}$, then $\mathcal{M}$ reduces U by another application of lemma 1.61. So $a = 1$ and $Se_o = \eta_o$. Therefore if S satisfies the condition of the theorem, we have $S(e_o - Pe_o) = \eta_o - Q\eta_o$ or $S\xi_o = \zeta_o$.

Again $SUS^{-1} = U^*$ implies that $SU^nS^{-1} = U^{*n}$, $n \geq 1$; $SU^nS^{-1}\zeta_o = U^{*n}\zeta_o$ or $SU^n\xi_o = U^{*n}\zeta_o$. By this observation we must define $S(\sum_{i=1}^{m} b_i U^i \xi_o) = \sum_{i=1}^{m} \overline{b}_i U^{*i}\zeta_o$. Since $(U^i\xi_o|U^j\xi_o) = (U^{*i}\zeta_o|U^{*j}\zeta_o) = \delta_{ij}$ for $0 \leq i,j < \infty$, we see that $(\xi_3|\xi_4) = (S\xi_3|S\xi_4)$ for $\xi_3,\xi_4 \in \mathcal{S}\{U^i\xi_o\}_{i=0}^{\infty}$. By linearity and continuity we extend the domain of definition of S from $\mathcal{L} \oplus \mathcal{S}\{U^i\xi_o\}_{i=0}^{\infty}$ to $\mathcal{L} \oplus \mathcal{F}\{U^i\xi_o\}_{i=0}^{\infty}$. It is clear that $SUS^{-1}\zeta = U^*\zeta$ for $\zeta \in \mathcal{S}\{U^{*i}\zeta_o\}_{i=0}^{\infty}$ and hence for ζ in the closure of this span. So $SUS^{-1} = U^*$ on $\mathcal{M} \oplus \mathcal{F}\{U^{*i}\zeta_o\}_{i=0}^{\infty} = \mathcal{K}$. In summary, $S\mathcal{L} = \mathcal{M}$; $S(\mathcal{F}\{U^i\xi_o\}_{i=0}^{\infty}) = \mathcal{F}\{U^{*i}\zeta_o\}_{i=0}^{\infty}$; S is anti-unitary; $S\eta_o = e_o$; $S^2\eta_o = Se_o = \eta_o$; and $SUS^{-1} = U^*$. So the theorem is proven in either case. We note that $S^2\eta_o = e_o$ is a consequence of the conditions $SUS^{-1} = U^*$ and $S\eta_o = e_o$. |

<u>Lemma 1.71</u> Suppose A is a linear operator with no non-trivial reducing space. Let S be unitary or anti-unitary such that $SAS^* = A^*$. Then $S^2 = c\cdot 1$ for some $c \in \mathcal{C}$.

Proof: $SAS^* = A^*$. Taking adjoints, we have $SA^*S^* = A$ or $S^*AS = A^* = SAS^*$. So $S^2A = AS^2$ and $S^2A^* = A^*S^2$. Since A has no non-trivial reducing space, the Double Commutant Theorem implies that S^2 is a scalar multiple of the identity; see p.448, [3]. |

<u>Lemma 1.72</u> If S is anti-unitary such that $S^2 = c\cdot 1$ where $c \in \mathcal{C}$, then $c = 1$ or $c = -1$.

Proof: $S^2 = c\cdot 1$ implies that $S^2S^* = cS^*$ or $S = cS^*$. Taking adjoints, we have $S^* = cS$. So $S = cS^* = \overline{c}S^*$. Thus $c - \overline{c} = 0$. Since $|c| = 1$ and c is a real number, we have $c = 1$ or $c = -1$. |

Corollary 1.73 If S is an anti-unitary operator satisfying the conditions of theorem 1.7, then $S^2 = 1$.

Proof: Lemma 1.71 implies that S^2 is a multiple of the identity. Lemma 1.72 implies that $S^2 = 1$ or $S^2 = -1$. $S^2\eta_o = \eta_o$ implies that $S^2 = 1$. |

Theorem 1.80 Let U be a partial isometry on $\mathcal{H}$ such that U has no non-trivial reducing space and $\dim \mathcal{N}_U = 1 \leq \dim \mathcal{N}_{U^*} < \infty$. Let $\zeta_o \in \mathcal{H}$ such that $\| U^{*i}\zeta_o \| = 1$ for $0 \leq i < \infty$. Then there exists $\xi_o \in \mathcal{H}$ such that $\| U^i\xi_o \| = 1$ for $0 \leq i < \infty$.

Proof: Let $\mathcal{N}_U = \mathcal{S}\{e_o\}$. If $\dim \mathcal{N}_{U^*} = 1$, then the theorem is true by lemma 1.41 and theorem 1.6 . We proceed by induction, assuming the theorem true for $\dim \mathcal{N}_{U^*} = n-1$ and proving it to hold in case $\dim \mathcal{N}_{U^*} = n$. Let $\mathcal{N}_{U^*}$ have the orthonormal basis $\{\eta_i\}_{i=1}^{n}$, where $\eta_i \perp e_o$ for $2 \leq i \leq n$. Let $\mathcal{K} = \mathcal{J}\{U^i\eta_1\}_{i=0}^{\infty}$. $\mathcal{K} \perp \mathcal{S}\{\eta_i\}_{i=2}^{n}$ since $\mathcal{R}_U \perp \mathcal{N}_{U^*}$. If $e_o \in \mathcal{K}$ or if $e_o \in \mathcal{K}^{\perp}$, then $\mathcal{K}$ is a reducing space by lemma 1.61 . Since $\| U^{*i}\zeta_o \| = 1$ for $0 \leq i < \infty$, we have $U^{*i}\zeta_o \perp \mathcal{N}_{U^*}$ or $(U^{*i}\zeta_o|\eta_1) = 0 = (\zeta_o|U^i\eta_1)$ for $0 \leq i < \infty$. So $\zeta_o \in \mathcal{K}^{\perp}$. Let P be the projection onto $\mathcal{K}$, and let $\theta_o = \| e_o - Pe_o \|^{-1}[e_o - Pe_o]$. Let $\mathcal{L} = \mathcal{K} + \mathcal{S}\{e_o\}$. $\mathcal{L} = \mathcal{K} \oplus \mathcal{S}\{\theta_o\}$. $\mathcal{K} = \mathcal{S}\{\eta_1\} \oplus U\mathcal{K}$. $U\mathcal{K} = U\mathcal{L}$. Since $\mathcal{N}_U \subset \mathcal{L}$, we have $U\mathcal{L} = U(\mathcal{L} \cap \mathcal{D}_U)$. Thus $\mathcal{L} = \mathcal{S}\{\eta_1\} \oplus U(\mathcal{L} \cap \mathcal{D}_U) \oplus \mathcal{S}\{\theta_o\}$. $U\mathcal{L} \perp \theta_o$ implies that $\mathcal{L} \perp U^*\theta_o$. $U^*\mathcal{K} = U^*(\mathcal{L} \ominus \mathcal{S}\{\theta_o\}) = U^*(\mathcal{S}\{\eta_1\} \oplus U(\mathcal{L} \cap \mathcal{D}_U)) = \mathcal{L} \cap \mathcal{D}_U$; or $U^*\mathcal{K} = \mathcal{L} \cap \mathcal{D}_U$. So $U^*\mathcal{K} \perp \mathcal{L}^{\perp}$ or $\mathcal{K} \perp U\mathcal{L}^{\perp}$. On $\mathcal{K}^{\perp}$, we define $\widetilde{U}$ a partial isometry as follows: $\widetilde{U}\theta_o = 0$, $\widetilde{U}\xi = U\xi$ for $\xi \in (\mathcal{K}^{\perp} \ominus \mathcal{S}\{\theta_o\}) = \mathcal{L}^{\perp}$. Since $\mathcal{N}_U \subset \mathcal{L}$, we see that $\mathcal{N}_{\widetilde{U}} = \mathcal{S}\{\theta_o\}$ and

$\mathcal{D}_{\tilde{U}} = \mathcal{L}^{\perp}$. We claim that $\tilde{U}^* = U^*$ on $\mathcal{K}^{\perp}$ and that $\tilde{U}$ has no non-trivial reducing space as an operator on $\mathcal{K}^{\perp}$. $\mathcal{K}^{\perp} = \mathcal{S}\{\theta_o\} \oplus \mathcal{L}^{\perp}$. We recall $U\mathcal{L} \subset \mathcal{L}$ or $U^*\mathcal{L}^{\perp} \subset \mathcal{L}^{\perp}$. But $U^*\theta_o \perp \mathcal{L}$. So $U^*\mathcal{K}^{\perp} = U^*(\mathcal{S}\{\theta_o\} \oplus \mathcal{L}^{\perp}) \subset \mathcal{L}^{\perp}$. Now let $\xi_1, \xi_2 \in \mathcal{K}^{\perp}$. We write $\xi_2 = a\theta_o + \xi_2'$, where $\xi_2' \in \mathcal{L}^{\perp}$. $(\tilde{U}^*\xi_1|\xi_2) = (\xi_1|\tilde{U}\xi_2) = (\xi_1|\tilde{U}[a\theta_o + \xi_2']) = (\xi_1|\tilde{U}\xi_2') = (\xi_1|U\xi_2') = (U^*\xi_1|\xi_2') = (U^*\xi_1|a\theta_o) + (U^*\xi_1|\xi_2') = (U^*\xi_1|\xi_2)$. So $\tilde{U}^* = U^*$ on $\mathcal{K}^{\perp}$. Now we suppose $\tilde{U}$ has a non-trivial reducing space $\mathcal{M}$ contained in but not equal to $\mathcal{K}^{\perp}$. If $\tilde{U}^*\tilde{U}\mathcal{M} = \mathcal{M}$, we have $\theta_o \perp \mathcal{M}$. On $\mathcal{K}^{\perp} \ominus \mathcal{S}\{\theta_o\}$, $U = \tilde{U}$ and $U^* = \tilde{U}^*$. Thus $\mathcal{M}$ reduces U, a contradiction. If $\tilde{U}^*\tilde{U}\mathcal{M}$ is properly contained in $\mathcal{M}$, then we have $\mathcal{M} \ominus \tilde{U}^*\tilde{U}\mathcal{M} = \mathcal{S}\{\theta_o\}$, since $\tilde{U}^*\tilde{U}$ is the projection onto $\mathcal{D}_{\tilde{U}}$. In this case we claim $\mathcal{K} \oplus \mathcal{M}$ is a non-trivial reducing space for U. Clearly $\mathcal{K} \oplus \mathcal{M} \neq \mathcal{K}$. $\mathcal{K} \oplus \mathcal{M} = \mathcal{K} \oplus \mathcal{S}\{\theta_o\} \oplus (\mathcal{M} \ominus \mathcal{S}\{\theta_o\}$. $U\mathcal{K} \subset \mathcal{K}$; $U^*\mathcal{K} \subset \mathcal{K} \oplus \mathcal{S}\{\theta_o\} = \mathcal{L}$; also $U(\mathcal{M} \ominus \mathcal{S}\{\theta_o\}) = \tilde{U}(\mathcal{M} \ominus \mathcal{S}\{\theta_o\}) \subset \mathcal{M}$. $U^*\mathcal{M} = \tilde{U}^*\mathcal{M} \subset \mathcal{M}$. Finally $U\theta_o \in U\mathcal{L} \subset \mathcal{K}$. So $\mathcal{L} \oplus \mathcal{M}$ reduces U, a contradiction. Therefore $\tilde{U}$ has no non-trivial reducing space. $\mathcal{N}_{\tilde{U}} = \mathcal{S}\{\theta_o\}$; $\mathcal{N}_{\tilde{U}^*} = \mathcal{S}\{\eta_i\}_{i=2}^{n}$ since $\tilde{U}^* = U^*$ on $\mathcal{K}^{\perp}$ and $\mathcal{S}\{\eta_i\}_{i=2}^{n} \subset \mathcal{K}^{\perp}$. So dim $\mathcal{N}_{\tilde{U}} = 1$ and dim $\mathcal{N}_{\tilde{U}^*} = n-1$. Also, $\zeta_o \in \mathcal{K}^{\perp}$ and $\|\tilde{U}^*\zeta_o\| = \|U^*\zeta_o\| = 1$ for $0 \leq i < \infty$. By the induction hypothesis we have that there exists $\xi_o \in \mathcal{K}^{\perp}$ such that $\|\tilde{U}^i\xi_o\| = 1$ for $0 \leq i < \infty$. Thus $\tilde{U}^i\xi_o \perp \theta_o$ for $0 \leq i < \infty$. So $\|U^i\xi_o\| = 1$ for $0 \leq i < \infty$. |

Example 1.81 To illustrate an application of the above theorem, we construct a partial isometry L_3 having no non-trivial reducing space such that dim $\mathcal{N}_{L_3} = 1$, dim $\mathcal{N}_{L_3^*} = 2$, and there is a L_3^*-chain. We define L_3 as follows on $\mathcal{J}\{e_i\}_{i=-\infty}^{\infty} = \mathcal{H}$:

$$L_3e_i = e_{i+1} \text{ for } -\infty < i \leq 1$$
$$L_3e_2 = [\sqrt{2}]^{-1}[e_3+e_4]$$
$$L_3e_{2n+1} = e_{2n+3} \text{ for } 0 < n < \infty$$
$$L_3e_4 = e_6;\ L_3e_6 = [\sqrt{2}]^{-1}[e_8+e_{10}];\ L_3e_8 = 0$$
$$L_3e_{2n} = e_{2n+2} \text{ for } 5 \leq n < \infty.$$

So $\eta_{L_3} = \mathcal{J}\{e_8\}$; $\eta_{L_3^*} = \mathcal{J}\{[\sqrt{2}]^{-1}[e_3-e_4],\ [\sqrt{2}]^{-1}[e_8-e_{10}]\}$. To show that L_3 has no non-trivial reducing space, we prove that e_8 is cyclic for L_3 and L_3^*. Then we prove any non-zero reducing space for L_3 must contain e_8. The proof is so similar to that in earlier examples that it is not repeated here. We observe that $\{U^{*i}([\sqrt{2}]^{-1}[e_3+e_4])\}_{i=0}^{\infty}$ is a L_3^*-chain. $\{U^ie_{10}\}_{i=0}^{\infty}$ and $\{U^ie_3\}_{i=0}^{\infty}$ are both L_3-chains; the existence of at least one L_3-chain is proven by theorems 1.80 and 1.6 .

We do not know if there is a partial isometry U satisfying the conditions of theorem 1.80 with the added requirement that dim $\eta_{U^*} \geq 3$.

Now we make an estimate which relates the number of pairwise orthogonal U*-chains to dim η_U, where U is a partial isometry having no non-trivial reducing space.

Lemma 1.85 Let $\mathcal{K}$ be a subspace of the Hilbert space $\mathcal{H}$, and let $\mathcal{L}$ be a n-dimensional subspace of $\mathcal{H}$. Then we have that dim $\{(\mathcal{K}+\mathcal{L})\ominus\mathcal{K}\} \leq n$.

Proof: Let $\xi \in \mathcal{L}$, $\zeta \in \mathcal{K}$. Then $\xi+\zeta\perp\mathcal{K}$ is equivalent to $-P\xi = P\zeta$ where P is the projection operator onto $\mathcal{K}$. $\zeta \in \mathcal{K}$ is equivalent to $P\zeta = \zeta$. So $\xi+\zeta \perp \mathcal{K}$ is equivalent to $\zeta = -P\xi$. Let $\{\xi_i\}_{i=1}^{\infty}$ be a linear basis for $\mathcal{L}$. Then $\{\xi_i - P\xi_i\}_{i=1}^{n}$ is a spanning set for $(\mathcal{K}+\mathcal{L})\ominus\mathcal{K}$. Thus $\dim[(\mathcal{K}+\mathcal{L})\ominus\mathcal{K}] \leq n.$ |

Theorem 1.90 Let U be a partial isometry on $\mathcal{H}$ such that U has no non-trivial reducing space and $1 \leq \dim \mathcal{N}_U < \infty$. Then there is a set D of vectors in $\mathcal{H}$ such that:

1) cardinality of $D \leq \dim \mathcal{N}_U$. If D is empty, we write card D = 0.
2) $(U^{*k}\xi_i | U^{*m}\xi_j) = \delta_{ij}\,\delta_{km}$ for $0 \leq k,m < \infty$ and $\xi_i, \xi_j \in D$ where $\xi_i \neq \xi_j$ for $i \neq j$.
3) If $\xi_0 \in \mathcal{H}$ such that $\|U^{*p}\xi_0\| = 1$ for $0 \leq p < \infty$, then $\xi_0 \in \sum_{i=1}^{\text{card } D} \oplus \mathcal{J}\{U^{*k}\xi_i\}_{k=0}^{\infty}$

Proof: Let $\mathcal{K} = \mathcal{J}\{U^i\mathcal{N}_{U^*}\}_{i=0}^{\infty}$. If ξ_0 is such that $U^{*p}\xi_0 \perp \mathcal{N}_{U^*}$ for $0 \leq p < \infty$, then $\xi_0 \perp \mathcal{K}$. $\mathcal{K} = \mathcal{N}_{U^*} \oplus U\mathcal{K}$. $\mathcal{K}$ is invariant under U. Let $\mathcal{L} = \mathcal{K} + \mathcal{N}_U$. $U\mathcal{L} = U\mathcal{K} \subset \mathcal{K}$. So $\mathcal{K} = \mathcal{N}_{U^*} \oplus U\mathcal{L}$. Let $\mathcal{J} = [(\mathcal{N}_{U^*} \oplus U\mathcal{L}) + \mathcal{N}_U] \ominus [\mathcal{N}_{U^*} \oplus U\mathcal{L}] = \mathcal{L} \ominus \mathcal{K}$. By lemma 1.85, we have $\dim \mathcal{J} \leq \dim \mathcal{N}_U$. $\mathcal{J} \perp \mathcal{K}$ implies $\mathcal{J} \perp \mathcal{N}_{U^*}$ or $\mathcal{J} \subset \mathcal{D}_{U^*}$. $U\mathcal{L} \subset \mathcal{K}$ implies $U\mathcal{L} \perp \mathcal{J}$ or $\mathcal{L} \perp U^*\mathcal{J}$. In fact, $U^{*t}\mathcal{J} \perp \mathcal{L}$ for $t \geq 0$ since $\mathcal{L}$ is invariant under U^*. If $\mathcal{J} \neq \{0\}$, then let $D = \{\xi_i\}_{i=1}^{s}$, an orthonormal basis for $\mathcal{J}$. We see that condition 1) is satisfied. We suppose that $(U^{*k}\xi_i | U^{*m}\xi_j) \neq 0$. We can assume $k \geq m$. $(U^{*k-m}\xi_i | \xi_j) \neq 0$. But $k-m > 0$ implies that $U^{*k-m}\xi_i \perp \mathcal{L}$. So $k-m = 0$. $(\xi_i | \xi_j) \neq 0$ implies that $i = j$. So 2) is satisfied. $\mathcal{K} \oplus (\sum_{i=1}^{s} \oplus \mathcal{J}\{U^{*j}\xi_i\}_{j=0}^{\infty})$ is invariant under U and U^*, as in earlier arguments. In the case $\mathcal{N}_U \subset \mathcal{K}$ or $\mathcal{J} = \{0\}$, we have $\mathcal{K} = \mathcal{H}$ by lemma 1.61 . In this situation where D is the empty set, we interpret $\sum_{i=1}^{s} \oplus \mathscr{I}\{U^{*j}\xi_i\}_{j=0}^{\infty} = \{0\}$. So in any case $\sum_{i=1}^{s} \oplus \mathscr{I}\{U^{*j}\xi_i\}_{j=0}^{\infty} = \mathcal{K}$. Hence 3) holds.|

By theorem 1.80 we know that a partial isometry U satisfying dim $\mathcal{N}_U = 1 =$ dim $\mathcal{N}_{U^*}$, having a U*-chain, and without non-trivial reducing spaces must have a U-chain. A natural question is whether the U-chain or a U-chain must be orthogonal to a U*-chain. The following example is best compared with example 1.2, L_1 .

Example 1.91 The following example will be constructed in several steps. We first construct a partial isometry in the following way. Let $\mathcal{K}$ be a Hilbert space with orthonormal basis $\{e_i\}_{i=0}^{\infty}$. Let $\{\xi_i\}_{i=1}^{\infty}$ be the skewed basis with respect to the orthonormal set $\{e_1, e_2\} \cup \{e_{2i+1}\}_{i=2}^{\infty}$. Similarly, let $\{\theta_i\}_{i=1}^{\infty}$ be the skewed basis with respect to the orthonormal set $\{e_3, e_4\} \cup \{e_{2i}\}_{i=3}^{\infty}$. That is, $\xi_1 = [\sqrt{2}]^{-1}[e_1+e_2]$; $\xi_2 = [\sqrt{2}]^{-2}[e_1-e_2] + [\sqrt{2}]^{-1}e_5$; ... . Also, $\theta_1 = [\sqrt{2}]^{-1}[e_3+e_4]$; $\theta_2 = [\sqrt{2}]^{-2}[e_3-e_4] + [\sqrt{2}]^{-1}e_6$; ... We define U* as follows :

$$U^*e_o = \xi_1;\ U^*e_1 = \theta_1;\ U^*e_3 = \xi_2;\ U^*e_5 = \theta_2$$

$$U^*e_{2n} = \xi_n \text{ for } n \geq 3;\ U^*e_{2n+1} = \theta_n \text{ for } n \geq 3$$

So $\mathcal{N}_{U^*} = \mathscr{I}\{e_2, e_4\}$; $\mathcal{N}_U = \mathcal{R}_{U^*} = \mathscr{I}\{e_o\}$. It is clear that e_o is cyclic for U* and that each non-zero reducing space for U must contain $\mathcal{N}_U$. So U has no non-trivial reducing space. Given $\theta \in \mathcal{K}$ with $\|\theta\| = 1$, we can easily obtain an integer k such that $\|U^k\theta\| < 1$. Loosely said, U moves each vector toward $\mathcal{N}_U$. So there is no U-chain. By theorem 1.80, there is no U*- chain.

Now we proceed to the next stage of the construction. Let $\mathcal{K} = \mathcal{J}\{U^i e_4\}_{i=0}^{\infty}$. Since $\mathcal{R}_U \perp \mathcal{N}_{U*}$, we have $e_2 \perp \mathcal{K}$. By lemma 1.61, if $e_o \in \mathcal{K}$ or if $e_o \perp \mathcal{K}$, then $\mathcal{K}$ reduces U. Note that $(U^2 e_4 | e_o) \neq 0$. Let $\mathcal{L} = \mathcal{K} + \mathcal{J}\{e_o\} = \mathcal{K} \oplus \mathcal{J}\{e_o - Pe_o\}$, where P is the projection onto $\mathcal{K}$.

Now let V denote the restriction of U to $\mathcal{L}$. Since $\mathcal{N}_U \subset \mathcal{L}$, we have that V is a partial isometry on $\mathcal{L}$. $\mathcal{N}_V = \mathcal{J}\{e_o\}$. $\mathcal{N}_{V*} = \mathcal{L} \ominus V\mathcal{L} = \mathcal{L} \ominus U\mathcal{L} = \mathcal{J}\{e_4, e_o - Pe_o\}$. We claim that V has no non-trivial reducing space. A reducing space for V must contain e_o, or else its orthogonal complement must; for $\mathcal{N}_V$ is one-dimensional. So let $\mathcal{L}'$ be a reducing space for V containing e_o. Then $\mathcal{L}'^{\perp}$ is invariant under V and hence under U. But this contradicts the fact that there is no U-chain, after an application of theorem 1.6 . We also claim that there is no V-chain. This is clear, since a V-chain would be a U-chain. Therefore, by theorem 1.80 there is no V*-chain.

Now we arrive at the last step of the construction. Let $\xi_o = \| e_o - Pe_o \|^{-1} [e_o - Pe_o]$. Let $\{\eta_i\}_{i=1}^{\infty}$ be an orthonormal set of vectors orthogonal to $\mathcal{K}$ and hence orthogonal to $\mathcal{L}$. We define our operator L_4 on $\mathcal{L} \oplus \mathcal{J}\{\eta_i\}_{i=1}^{\infty}$ as follows:

The restrictions of L_4, V, and U to $\mathcal{L}$ are the same.

$L_4 \eta_1 = \xi_o$; $L_4 \eta_i = \eta_{i-1}$ for $i \geq 2$.

So $\mathcal{N}_{L_4} = \mathcal{J}\{e_o\}$; $\mathcal{N}_{L_4^*} = \mathcal{J}\{e_4\}$. By an argument analogous to the one just employed to demonstrate that V has no non-trivial reducing space, we see that L_4 has no non-trivial reducing space.

We have a L_4^*-chain, namely $\{L_4^{*i}\xi_o\}_{i=0}^{\infty}$. So by theorems 1.80 and 1.6, there is a L_4-chain. If ζ is a vector in a L_4-chain, then $(\zeta|\eta_{k_i}) \neq 0$ for a co-final set of integers $\{i_k\}$. That is, if $(\zeta|\eta_m) = 0$ for all m greater than some fixed integer M, then $\{L_4^{m}\zeta\}_{m=M+1}^{\infty}$ is a L_4-chain contained in $\mathcal{L}$. But $L_4 = V$ on $\mathcal{L}$, yielding a V-chain and a contradiction. Thus we find that no L_4-chain is orthogonal to the L_4^*- chain $\{L_4^{*i}\xi_o\}_{i=0}^{\infty}$. By theorem 1.90 $\{L_4^{*i}\xi_o\}_{i=0}^{\infty}$ or rather the span of this L_4^*-chain contains each L_4^*-chain, since $\mathcal{J}\{L_4^{i}\eta_{L_4^*}\}_{i=0}^{\infty}$ $= \mathcal{K} = \mathcal{J}\{U^{i}e_4\}_{i=0}^{\infty}$. In other words, the span of the L_4^*-chain which contains the span of every other L_4^*-chain need not be orthogonal to the span of the corresponding L_4- chain or to the span of any L_4-chain.|

We have only partially answered the question we posed before presenting the above example. It is not clear whether a given L_4-chain must be orthogonal to at least one L_4^*-chain. In example 1.91, the construction of the partial isometry U in the first stage can be used to build partial isometries such that the difference in dimension of the null space of the operator with that of the null space of its adjoint is a given integer. In fact , the operator U on $\mathcal{K}$ in 1.91 might be called a U*-chain of defect two, since e_o is cyclic for U* and dim $\eta_{U^*} = 2$.

Other examples may be constructed by minor modification of those examples given. For a partial isometry U, one might ask what significance the orthogonality of $\mathcal{N}_U$ and $\mathcal{N}_{U*}$ might have. If $\mathcal{N}_U \perp \mathcal{N}_{U*}$, then U*U commutes with UU*. In example 1.2 we see that $\mathcal{N}_{L_1}$ is not orthogonal to $\mathcal{N}_{L_1^*}$. We now define a partial isometry S_1 which is a "finite-dimensional" modification of L_1 such that $\mathcal{N}_{S_1} \perp \mathcal{N}_{S_1^*}$. Let $[\{e_i\}_{i=-\infty}^{\infty} \cup \{\eta_o, \eta_1\}]$ be an orthonormal basis in $\mathcal{H}$. Let $S_1 e_j = e_{j+1}$ for $-\infty < j < 0$ and $1 < j < \infty$. Let $S_1 e_o = [\sqrt{2}]^{-1}[e_1+e_2]$; let $S_1\eta_o = \eta_1$; let $S_1\eta_1 = [\sqrt{2}]^{-1}[e_1-e_2]$. So $\mathcal{N}_{S_1} = \mathcal{S}\{e_1\}$ and $\mathcal{N}_{S_1^*} = \mathcal{S}\{\eta_o\}$. Thus we see that at least in simple cases the orthogonality of null spaces seems to be of no consequence.

We conclude this section with three more examples. First, we exhibit L_5 having no non-trivial reducing space and dim $\mathcal{N}_{L_5}$ = dim $\mathcal{N}_{L_5^*} = \infty$. Secondly, we have L_6 having again no non-trivial reducing space and dim $\mathcal{N}_{L_6} = 1$, dim $\mathcal{N}_{L_6^*} = \infty$. So the finite dimensionality of the null space of a partial isometry together with the hypothesis of no non-trivial reducing space need not imply the finite dimensionality of the null space of its adjoint operator. Our final example is L_7; L_7 satisfies the hypothesis of theorem 1.7 with the exception that dim $\mathcal{N}_{L_7}$ = dim $\mathcal{N}_{L_7^*} = 2$. However, L_7 is not anti-unitarily equivalent to its adjoint.

Example 1.92 Let $\mathcal{H}$ be a Hilbert space with orthonormal basis $\{e_i\}_{i=1}^{\infty}$. Let $\eta_1 = e_1$; let $[\sqrt{2}]^{-1}[e_{2i}-e_{2i+1}] = \eta_{2i}$ for $1 \leq i \leq \infty$; let $[\sqrt{2}]^{-1}[e_{2i}+e_{2i+1}]$ for $1 \leq i < \infty$. Let L_5 be the partial isometry with domain space $\mathcal{S}\{\eta_{2i+1}\}_{i=0}^{\infty}$, with range space $\mathcal{V}\{e_{2i}\}_{i=1}^{\infty}$, and defined as follows: $L_5\eta_{2i+1} = e_{2i+1}$ for $0 \leq i < \infty$. So $\eta_{L_5} = \mathcal{V}\{\eta_{2i}\}_{i=1}^{\infty}$; $\eta_{L_5^*} = \mathcal{V}\{e_{2i+1}\}_{i=0}^{\infty}$. It is clear that e_1 is cyclic for L_5. Loosely said, L_5^* moves vectors toward $\eta_1 = e_1$. So a reducing space for L_5 contains a vector of the form $\sum_{n=0}^{\infty} a_{2n+1}\eta_{2n+1}$ with $a_1 \neq 0$. If we designate such a vector as ξ_o, then we have $\xi_o - 2\, L_5^* L_5^2 L_5^* \xi_o = a_1 e_1$. So L_5 has no non-trivial reducing space, $\dim \eta_{L_5} = \dim \eta_{L_5^*} = \infty$.|

Example 1.93 Let $\mathcal{H}$ be a Hilbert space with the orthonormal basis $\{e_i\}_{i=1}^{\infty} \cup [\{\eta_k^j\}_{k=1}^{\infty}]_{j=1}^{\infty}$. Let $\theta = \sum_{i=1}^{\infty} [\sqrt{2}]^{-i} e_i$. Let $L_6 =$ be defined as follows:

$L_6\theta = 0$

$L_6\eta_j^j = e_j$ for $1 \leq j < \infty$.

$L_6\eta_k^j = \eta_{k+1}^j$ for all (j,k) such that $j \neq k$, $0 < j,k < \infty$.

Let $\xi_i' = \| e_i-(e_i|\theta)\theta \|^{-1}[e_i-(e_i|\theta)\theta]$ for $1 \leq i < \infty$.

Let $\{\xi_i\}_{i=1}^{\infty}$ be the Gramm orthonormalization of $\{\xi_i'\}_{i=1}^{\infty}$.

$L_6\xi_i = \eta_{i+1}^i$ for $1 \leq i < \infty$.

So $\eta_{L_6} = \mathcal{I}\{\theta\}$; $\eta_{L_6^*} = \mathcal{V}\{\eta_1^j\}_{j=1}^{\infty}$. We note that $(\xi_i'|e_j) \neq 0$ for $1 \leq i,j < \infty$. We will sketch a proof that L_6 has no non-trivial reducing space. It is enough to show that a non-zero reducing space for L_6 must contain $\mathcal{V}\{\eta_1^j\}_{j=1}^{\infty} = \eta_{L_6^*}$, since $\eta_{L_6^*}$ is cyclic for L_6. It can be shown that if $\xi \in \mathcal{H}$, $\xi \neq 0$, then there is a polynomial $P(x,y)$ such that $P(L_6,L_6^*)\xi \neq 0$, $P(L_6,L_6^*)\xi \notin \mathcal{I}\{\theta\}$, and $(P(L_6,L_6^*)\xi|\theta) \neq 0$. If we let $\xi_1 = P(L_6,L_6^*)\xi$, then we can write $\xi_1 - L_6^* L_6 \xi_1 = \sum_{i=m}^{\infty} a_i e_i$ where $a_m \neq 0$. But $L_6^{*m}(\sum_{i=m}^{\infty} a_i e_i) - L_6 L_6^{*m+1}(\sum_{i=m}^{\infty} a_i e_i) = a_m \eta_1^m$. Now

$L_6^m \eta_1^m = e_m$. $L_6^* L_6 e_m = c\xi_m'$ with c a non-zero complex number. $\xi_m' = \sum_{i=1}^{\infty} b_i e_i$, $b_i \neq 0$ for $1 \leq i < \infty$. So by a process similar to the one used in obtaining η_1^m, we can now obtain η_1^j for $1 \leq j < \infty$. Thus L_6 has no non-trivial reducing space.|

Example 1.94 Let $\mathcal{H}$ be a Hilbert space with orthonormal basis $\{e_i\}_{i=-\infty}^{\infty}$. Let $\{\theta_i\}_{i=-\infty}^{-5}$ be the skewed basis with respect to the orthonormal set $\{e_i\}_{i=-\infty}^{-5}$. Let $\{\eta_i\}_{i=4}^{\infty}$ be the skewed basis with respect to the orthonormal set $\{e_i\}_{i=4}^{\infty}$. Let L_7 be defined as follows:

$L_7\theta_i = e_i$ for $-\infty < i < -5$; $L_7\theta_{-5} = e_{-4}$; $L_7 e_i = e_{i+1}$ for $-4 \leq i \leq -1$; $L_7 e_0 = [\sqrt{2}]^{-1}[e_1 + e_2]$; $L_7 e_1 = 0$; $L_7 e_2 = e_3$; $L_7 e_3 = \eta_4$; $L_7 e_4 = 0$; $L_7 e_i = \eta_i$ for $5 \leq i < \infty$.

We sketch a proof that L_7 has no non-trivial reducing space. We suppose that $\mathcal{K}$, a non-zero subspace, reduces L_7. If we choose a non-zero vector in $\mathcal{K}$, then either the vector is not orthogonal to $\mathcal{N}_{L_7}$ or there is a polynomial $P(x,y)$ with non-commuting variables such that $P(L_7, L_7^*)$ applied to the vector is not orthogonal to $\mathcal{N}_{L_7}$. So $\mathcal{K}$ contains a non-zero vector ξ of the form $ae_1 + be_4$. If $a = 0$, then $(L_7 L_7^* \xi | e_1) \neq 0$. So we can assume $a \neq 0$. We project ξ onto $\mathcal{N}_{L_7^*}$, obtaining a non-zero multiple of $[\sqrt{2}]^{-1}[e_1 - e_2]$. We project this vector back onto $\mathcal{N}_{L_7}$ to obtain a non-zero multiple of e_1. But e_1 is cyclic for L_7 and L_7^*. Hence $\mathcal{K} = \mathcal{H}$.

We suppose S is anti-unitary such that $SL_7S^* = L_7^*$. As before, we have $S^*L_7S = L_7^*$, $L_7 = S^*L_7^*S$, and $L_7^n = S^*L_7^{*n}S$ for

$n \geq 0$. Since $S^*L_7S = L_7^*$, we have $S\mathcal{N}_{L_7^*} \subset \mathcal{N}_{L_7}$. We let $Se_{-5} = \zeta \in \mathcal{N}_{L_7}$, $\| \zeta \| = 1$. $\| L_7^k e_{-5} \| = 1$ for $0 \leq k \leq 6$. $\| L_7^{*m} \zeta \| < 1$ for some m, $1 \leq m \leq 3$, by direct verification using the fact that $\zeta \in \mathcal{N}_{L_7}$. But $L_7^m e_{-5} = S^*L_7^*\zeta$, $1 \leq m \leq 3$. So we have a contradiction. Hence L_7 is not anti-unitarily equivalent to L_7^*. |

We now consider a few consequences of an operator's being anti-unitarily equivalent to its own adjoint. First we note that every normal operator is anti-unitarily equivalent to its own adjoint. In fact, if N is a normal operator on $\mathcal{H}$, $\mathcal{A}$ is a maximal commutative symmetric ring of operators containing N, and ξ_o is a cyclic vector for $\mathcal{A}$, then we have $\mathcal{H}$ is isomorphic to $L^2(M, \mu)$ and $\mathcal{A}$ is isometric-isomorphic to $C(M)$, where M is the maximal ideal space of $\mathcal{A}$. Thus if U is the conjugate operator which maps the pre-image of $\xi(m)$ to the pre-image of $\overline{\xi(m)}$ for $\xi(m) \in L^2(M,\mu)$, then it is clear that $UNU^* = N^*$, since $\overline{N(m)} = \widehat{N^*}(m)$. In view of the fact that the class of operators anti-unitarily equivalent to their adjoints includes the normal operators, one might expect that operators in this class have some properties in common with normal operators.

Lemma 1.95 Let A be a linear operator anti-unitarily equivalent to A^*. Then the spectrum of A is the approximate point spectrum of A.

Proof: Let U be an anti-unitary operator such that $UAU^* = A^*$.

The approximate point spectrum of an operator is contained in its spectrum. We suppose λ is a complex number which is not in the approximate point spectrum of A. Then there is $\alpha > 0$ such that $\|(A-\lambda 1)\xi\| \geq \alpha\|\xi\|$ for $\xi \in \mathcal{H}$. We claim that $\mathcal{R}_{A-\lambda 1}$ is dense in $\mathcal{H}$. If $([A-\lambda 1]\xi|\zeta) = 0$ for all $\xi \in \mathcal{H}$, then $[A^*-\bar{\lambda}1]\zeta = 0$. Now $\|(A^*-\bar{\lambda}1)\zeta\| = \|(UAU^*-U\lambda U^*)\zeta\| = \|(A-\lambda 1)U^*\zeta\| = 0$. So $U^*\zeta = 0 = \zeta$. Hence if λ is not an approximate eigenvalue, then $(A-\lambda 1)^{-1}$ exists.|

<u>Lemma 1.96</u> Let $\sigma(A)$ denote the spectrum of A, where A is a linear operator. If there is an anti-unitary operator V such that $VAV^* = A^*$, then $\sigma(A) = \overline{\sigma(A^*)}$ and A^*A is unitarily equivalent to AA^*.

Proof: $VAV^* = A^*$; $V(A-\bar{\lambda}1)V^* = A^*-\lambda 1$. So the existence of $(A-\bar{\lambda}1)^{-1}$ is equivalent to the existence of $(A^*-\lambda 1)^{-1}$. Hence $\sigma(A) = \overline{\sigma(A^*)}$.

$VAV^* = A^*$ implies that $VA^*V^* = A$. $AA^* = VA^*AV^*$. As mentioned earlier, we can find U_1 an anti-unitary operator such that $U_1^* = U_1$ and $U_1AA^*U_1^* = AA^*$. So $AA^* = U_1AA^*U_1^* = U_1VA^*AV^*U_1^* = (U_1V)A^*A(U_1V)^*$. Since U_1V is the product of anti-unitary operators and thus unitary, the lemma holds.|

We are unable to state necessary and sufficient conditions for an operator to be anti-unitarily equivalent to its adjoint. If A is an operator which is anti-unitarily equivalent to A^* by the anti-unitary U, then (cA), $c \in \mathbb{C}$, is anti-unitarily equivalent to $(cA)^*$ by the same anti-unitary U. This nned not hold for unitary equivalence. We do not know whether the anti-unitary operator U giving the anti-unitary equivalence of an operator A to its adjoint A* can always be taken so that U=U*, or, what is the same, so that $U^2 = 1$. Each anti-unitary operator U such that U = U* is given by conjugation of Fourier coefficients with respect to an orthonormal basis; there exists an orthonormal basis $\{e_i\}_{i=1}^{\infty}$ such that for $\sum_{i=1}^{\infty} a_i e_i \in \mathscr{H}$, we have $U(\sum_{i=1}^{\infty} a_i e_i) = \sum_{i=1}^{\infty} \overline{a}_i e_i$. Such an anti-unitary operator is called a conjugation; p.357-360, [9].

Lemma 1.97 Let A be a linear operator on $\mathscr{H}$; then A is anti-unitarily equivalent to A* by a conjugation if and only if with respect to some basis $\{e_i\}_{i=1}^{\infty}$ the matrix (a_{ij}) of A satisfies $a_{ij} = a_{ji}$ for $1 \leq i,j < \infty$.

Proof: We suppose UAU* = A*, with U anti-unitary and U = U*. Let $\{e_i\}_{i=1}^{\infty}$ be an orthonormal basis for $\mathscr{H}$ with respect to which U is conjugation of Fourier coefficients. $UAU^*e_j = UAe_j = U(\sum_{i=1}^{\infty} a_{ij}e_i) = \sum_{i=1}^{\infty} \overline{a}_{ij}e_i$. So the matrix of UAU* is $(\overline{a}_{ij})$. But the matrix of A* is $(\overline{a}_{ji})$. Thus $a_{ij} = a_{ji}$ for $1 \leq i,j < \infty$.

We suppose now that the matrix (a_{ij}) of the operator A with respect to the basis $\{e_i\}_{i=1}^{\infty}$ is such that $a_{ij} = a_{ji}$ for $1 \leq i,j < \infty$. Let U be the anti-unitary operator given by conjugation of Fourier coefficients with respect to $\{e_i\}_{i=1}^{\infty}$. By reversing the above computations we see that UAU* = A*. |

If H is a Hermitian operator, then with respect to some basis H has the matrix (h_{ij}) where $h_{ij} = h_{ji}$. Since the matrix of H* = H is $(\overline{h}_{ji})$, we see that h_{ij} is a real number for $1 \leq i,j < \infty$. Thus lemma 1.97 implies that a Hermitian operator has a real matrix with respect to some basis. If N is a normal operator, then N also satisfies the hypothesis of lemma 1.97 . So there is a basis with respect to which the matrix (b_{ij}) of N satisfies $b_{ij} = b_{ji}$. When we write out what it means for the matrices (b_{ij}) and $(\overline{b}_{ji})$ to commute, we have $\sum_k b_{ik}\overline{b}_{jk} = \sum_k \overline{b}_{ki} b_{kj} = \sum_k \overline{b}_{ik} b_{jk}$. So the inner product of each row of the matrix (b_{ij}) of N with each other row is a real number. Also, it is clear by reversing the above argument that each such matrix yields a normal operator.

CHAPTER II

NORMAL CONJUGATE OPERATORS

We now obtain a structure theorem on Hermitian conjugate operators. In Stone's pioneering book on transformations on Hilbert space, there is a structure theorem for a conjugation; p. 357-360, [9]. A conjugation J is an anti-unitary operator such that $J^2 = 1$. It is easy to see that $J^2 = 1$ is equivalent to $J = J^*$. A conjugation J is given by conjugation of Fourier coefficients with respect to a suitable orthonormal basis depending on J; or J can be represented as conjugation of functions on a suitable L^2-space. Stone's book contains interesting material on real linear operators, linear operators which commute with a conjugation operator.

Theorem 2.1 Let H be a Hermitian conjugate operator on $\mathcal{H}$. Then there exists a symmetric commutative ring of operators $\mathcal{A}$ such that $K \in \mathcal{A}$, $K = K^*$ implies that $KH = HK$; and $\mathcal{A}$ is maximal with respect to set inclusion in the class of all commutative symmetric subrings of $\mathcal{B}(\mathcal{H})$.

Proof: Let $\mathcal{A}'$ be a symmetric commutative ring maximal with respect to set inclusion in the class $\mathcal{D}$ of symmetric commutative rings such that $\mathcal{A}'' \in \mathcal{D}$, $K \in \mathcal{A}''$, $K = K^*$ imply that $KH = HK$ and $H^2 \in \mathcal{A}$. The ring generated by H^2 is an element of $\mathcal{D}$. Let $\mathcal{A}$ be a maximal commutative symmetric ring containing $\mathcal{A}'$. We will show that $\mathcal{A}' = \mathcal{A}$. We have that $\mathcal{H}$ is isomorphic to $L^2(M,\mu)$, where M is the maximal ideal space of $\mathcal{A}$ and μ is regular Borel measure corresponding to a fixed vector ξ_0 which is cyclic for $\mathcal{A}$. By construction, the mapping of ξ to $\xi(m)$ from $\mathcal{H}$ to $L^2(M,\mu)$ has the property that $\xi_0(m) = 1$ for all $m \in M$. Hence forward we identify $\mathcal{H}$ with $L^2(M,\mu)$. We define $U(\xi(m)) = \overline{\xi(m)}$. Clearly U is anti-unitary, $U^2 = 1$ or $U = U^*$. Since a Hermitian operator K in $\mathcal{A}$ has a real-valued Gelfand transform $\widehat{K(m)}$ in $C(M)$, we have that $UK = KU$ for $K \in \mathcal{A}$, $K = K^*$. We claim $UH \in \mathcal{A}'$. $UH^2 = H^2U$ since $H^2 \in \mathcal{A}'$ and H^2 is Hermitian. $(UH)^* = H^*U^* = HU$. So $UH(UH)^* = UH^2U = H^2U^2 = H^2$; and $(UH)^*(UH) = HU^2H = H^2$. Thus UH is a normal operator. We must show that the Hermitian parts of UH commute with H.

$UH = 2^{-1}[UH+HU] + i[2i]^{-1}[UH-HU]$

$2^{-1}[UH+HU]H = 2^{-1}[UH^2+HUH] = 2^{-1}[H^2U+HUH]$

$H\{2^{-1}[UH+HU]\} = 2^{-1}[H^2U+HUH]$

So H commutes with $2^{-1}[UH+HU]$.

Now, $H[2i]^{-1}[UH-HU] = -[2i]^{-1}[HUH-H^2U] = [2i]^{-1}[H^2U-HUH]$

and $[2i]^{-1}[UH-HU]H = [2i]^{-1}[UH^2-HUH] = [2i]^{-1}[H^2U-HUH]$

Thus $2^{-1}[UH+HU]$ and $[2i]^{-1}[UH-HU]$ commute with H. Now we show that UH commutes with $\mathcal{A}'$. Let $A \in \mathcal{A}'$; let $A = K_1 + iK_2$ where K_1 and K_2 are Hermitian elements of $\mathcal{A}'$. Since $\mathcal{A}' \subset \mathcal{A}$, we see $UK_i = K_iU$ for $i = 1,2$. $UHA(K_1+iK_2) = U(HK_1-iHK_2) = U(K_1-iK_2)H = (K_1+iK_2)UH = AUH$. So UH commutes with $\mathcal{A}'$. Hence, if we adjoin UH and $(UH)^*$ to $\mathcal{A}'$, then we obtain a ring in the class $\mathcal{B}$ from which $\mathcal{A}'$ was chosen. Since $\mathcal{A}'$ is maximal in $\mathcal{B}$, we have $UH \in \mathcal{A}'$.

We now show that $\mathcal{A}'$ is weakly closed. If $\{A_a\}$ is a net of operators converging in the weak topology for $\mathcal{B}(\mathcal{H})$ to an operator A, then $\{A_a^*\}$ converges weakly to A^*. Thus $\{2^{-1}[A_a+A_a^*]\}$ converges weakly to $2^{-1}[A+A^*]$, and $\{[2i]^{-1}[A_a-A_a^*]\}$ converges weakly to $[2i]^{-1}[A-A^*]$. Also, if $\{A_a\}$ is a net weakly converging to A, then $\{A_aH\}$ converges weakly to AH and $\{HA_a\}$ converges weakly to HA. If K is in the weak closure of $\mathcal{A}'$, then K commutes with each operator in $\mathcal{A}'$. The Hermitian parts of K are weakly approximated by Hermitian elements of $\mathcal{A}'$ which commute with H. Thus, by the above remarks on weak convergence, the Hermitian parts of K commute with H. So $\mathcal{A}'$ is weakly closed.

Now we show that ξ_o is a cyclic vector for $\mathcal{A}'$. Let $\mathcal{A}'[\xi_o] = \mathcal{H}_1$. Let P be the projection onto $\mathcal{H}_1$. Clearly P commutes with $\mathcal{A}'$. Since $\xi_o(m) = 1$ for all $m \in M$, we have $U\xi_o = \xi_o$. Sums of the form $\sum_{i=1}^{n} a_iA_i\xi_o$ with $A_i \in \mathcal{A}'$ and $A_i = A_i^*$ are dense in $\mathcal{H}_1$. We recall that $UA_i = A_iU$. Thus $U(\sum_{i=1}^{n} a_iA_i\xi_o) = \sum_{i=1}^{n} \bar{a}_iUA_i\xi_o = \sum_{i=1}^{n} \bar{a}_iA_iU\xi_o = \sum_{i=1}^{n} \bar{a}_iA_i\xi_o$. So $\mathcal{H}_1$ is invariant under U. $UH \in \mathcal{A}'$ implies that $UH\mathcal{H}_1 \subset \mathcal{H}_1$. Now $H = U(UH)$. So $H\mathcal{H}_1 \subset \mathcal{H}_1$ and $HP = PHP$. Taking adjoints, $P^*H^* = P^*H^*P^*$ or $PH = PHP$. So $PH = HP$. Again by the maximality of $\mathcal{A}'$ in the class $\mathcal{D}$, we have $P \in \mathcal{A}'$. So $P \in \mathcal{A}$. Now $1 \in \mathcal{A}'$. By definition of P, $(1-P)\xi_o = 0$ or $[1 - \widehat{P}(m)]\xi_o(m) = 0$ for almost all $m \in M$. Since $\widehat{P}(m)$ is continuous and $\xi_o(m) = 1$ for all $m \in M$, we have $\widehat{P}(m) = 1$ for all $m \in M$. So $P = 1$ and ξ_o is cyclic for $\mathcal{A}'$. Since $\mathcal{A}'$ is weakly closed, symmetric, and with a cyclic vector, we have that $\mathcal{A}'$ is maximal. So $\mathcal{A} = \mathcal{A}'$. |

Corollary 2.11 Let H be a Hermitian conjugate operator and let $\mathfrak{a}$ be the maximal commutative symmetric ring of theorem 2.1 . Let ξ_o be cyclic for $\mathfrak{a}$, and let U be conjugation of functions in $L^2(M,\mu)$ where M is the maximal ideal space of $\mathfrak{a}$ and μ is the regular Borel measure corresponding to ξ_o. Then UH and HU are in $\mathfrak{a}$, and H can be represented on $L^2(M,\mu)$ by $H\xi(m) = \widehat{(HU)}(m)\overline{\xi(m)}$ for $\xi \in \mathscr{H}$.

Proof: From theorem 2.1, we have UH and hence HU are in $\mathfrak{a}' = \mathfrak{a}$. $H = U(UH)$. So $(H\xi)(m) = (UUH\xi)(m) = U(UH\xi)(m) = \widehat{(UH)}(m)\overline{\xi(m)} = \widehat{(UH)}^*(m)\overline{\xi(m)} = \widehat{(HU)}(m)\overline{\xi(m)}$. |

We note that if U is a unitary or anti-unitary operator, then $U^2 = 1$ is equivalent to $U = U^*$. Now we turn to study conjugate normal operators. The following lemma is analogous to the familiar one concerning operators.

Lemma 2.2 Let N be a conjugate operator. Then $\| N^*\theta \| = \| N\theta \|$ for all $\theta \in \mathscr{H}$ is equivalent to N being normal.

Proof: Since N^*N and NN^* are linear operators, we have the usual polar decomposition of the inner product for $(N^*N\xi|\eta)$ and for $(NN^*\xi|\eta)$. Hence $N^*N = NN^*$ is equivalent to $(N^*N\theta|\theta) = (NN^*\theta|\theta)$ for all $\theta \in \mathscr{H}$. But this last equation is equivalent to $\| N\theta \|^2 = \| N^*\theta \|$. |

We have found no fruitful definition of spectrum for a conjugate operator. However, the next lemma gives a hint to the structure of a normal conjugate operator by noting a property of the spectrum of its square.

Lemma 2.3 If N is a normal conjugate operator, then the spectrum of N^2 is closed under conjugation; that is, $\sigma(N^2) = \overline{\sigma(N^2)}$.

Proof: N^2 is a normal operator, since $N^*N = NN^*$ implies $N^2N^{*2} = N^{*2}N^2$. Let $c \in \sigma(N^2)$. We know the spectrum of the normal operator N^2 is its approximate point spectrum. Let $\alpha > 0$. There exists non-zero $\xi \in \mathcal{H}$ such that $\| N^2\xi - c\xi \| < \alpha \|\xi\|$. Then $\| N^2(N\xi) - \overline{c}(N\xi) \| < \|N\| \cdot \alpha \cdot \|\xi\|$. So if $N\xi \neq 0$, then $\overline{c}$ is in the approximate point spectrum of N^2. But if $N\xi = 0$, then $N^2\xi = 0$. In case $c \neq 0$, by choice of $\alpha < |c|$ we have $N^2\xi \neq 0$ and hence $N\xi \neq 0$. If $c = 0$, then $\overline{c} \in \sigma(N^2)$. |

Before continuing a development of the structure of normal conjugate operators, we mention an easily obtained but gross result relating normal operators and normal conjugate operators. Let N be a normal (conjugate normal) operator. Then there is a conjugation U such that UN is conjugate normal (normal) and U commutes with N^*N. This is obtained by taking a conjugation U which commutes with N^*N.

In view of the fact that some of the computations to follow are rather long, we give two examples to which the reader can refer the machinery of our structure theorem, 2.8 .

Example 2.51 Let $\mathcal{H} = \mathcal{J}\{e_i\}_{i=1}^{\infty}$ where $e_i \perp e_j$ for $i \neq j$ and $\| e_i \| = 1$ for $1 \leq i < \infty$. We define U as follows:

$$Ue_{2i} = -e_{2i-1} \text{ for } 1 \leq i < \infty$$

$$Ue_{2i-1} = e_{2i} \text{ for } 1 \leq i < \infty$$

We extend U by conjugate linearity; $U(\sum_{i=1}^{\infty} a_i e_i) = -\sum_{i=1}^{\infty} \overline{a}_{2i} e_{2i-1} + \sum_{i=1}^{\infty} \overline{a}_{2i-1} e_{2i}$. So $U^2 = -1$ or $U = -U^*$. |

Example 2.52 Let $\mathcal{H}$ be a Hilbert space with orthonormal basis $\{\xi_n\}_{n=-\infty}^{\infty}$. Let $S(\xi_n) = \xi_{n+1}$ for $-\infty < n < \infty$, and extend by conjugate linearity so that $S(\sum_{-\infty}^{\infty} b_n \xi_n) = \sum_{-\infty}^{\infty} \overline{b}_{n-1} \xi_n$. We note that if we set $R\xi_n = \xi_{n+1}$ and extend by linearity, we obtain the familiar unitary operator, the bilateral shift, which can be represented by multiplication by $e^{i\theta}$ on $L^2([0,2\pi])$ since $e^{i\theta}e^{in\theta} = e^{i(n+1)\theta}$ and $\mathcal{V}\{e^{in\theta}\}_{n=-\infty}^{\infty} = L^2([0,2\pi])$. The results of theorem 2.8 are easily accessible in the case of S. To represent S, we let $\xi_n = e^{in\theta}$ for $-\infty < n < \infty$ and $\theta \in [0,2\pi]$; U denotes the conjugation of functions in $L^2([0,2\pi])$; finally, we let T be the mapping of $[0,2\pi]$ onto itself given by $T(\theta) = -\theta \mod 2\pi$. In the usual fashion we identify 0 with 2π and $[0,2\pi]$ with the unit circle in the complex plane. Then we see $S(\sum_{-\infty}^{\infty} b_n e^{in\theta}) = \sum_{-\infty}^{\infty} \overline{b}_{n-1} e^{in\theta} = e^{i\theta} \sum_{-\infty}^{\infty} \overline{b}_n e^{in\theta} = R(\sum_{-\infty}^{\infty} \overline{b}_n e^{in\theta}) = RU(\sum_{-\infty}^{\infty} b_n e^{-in\theta}) = RU(\sum_{-\infty}^{\infty} b_n e^{inT(\theta)})$. So if $\xi(\theta) \in L^2([0,2\pi])$, we can write $(S\xi)(\theta) = e^{i\theta}\overline{\xi\{T(\theta)\}}$. S is given by a rotation T, a conjugation U, and a multiplication R on $L^2([0,2\pi])$. We note that T^2 is the identity map on $[0,2\pi]$. Our structure theorem, 2.8, is simply a generalization from this example.

Lemma 2.6 Let N be a normal conjugate operator on $\mathcal{H}$; let $\mathcal{L} = \{\xi \mid N\xi = 0\}$. On $\mathcal{K} = \mathcal{H} \ominus \mathcal{L}$, we have that $N = U\sqrt{(N^*N)}$, where U is an anti-unitary operator which commutes with N and with N^*. Also, $\mathcal{K}$ reduces N.

Proof: By lemma 2.2, $\| N\xi \| = \| N^*\xi \|$ for all $\xi \in \mathcal{K}$. So $\mathcal{L}$ is the null-space of N^*. If $N^*N\xi = 0$ for some $\xi \in \mathcal{K}$, then $0 = (N^*N\xi | \xi) = (N\xi | N\xi)$; so $N\xi = 0$. Clearly $N\xi = 0$ implies that $N^*N\xi = 0$. So $\mathcal{L}$ is the null-space of N^*N and, by a similar argument, of NN^*. Since $\sqrt{(N^*N)}$ can be approximated in the norm by polynomials in N^*N with real coefficients, and since N and N^* both commute with N^*N, we have that $\sqrt{(N^*N)}$ commutes with N and with N^*. From this point on, we consider only the subspace $\mathcal{K}$, as is clearly sufficient.

If $\eta \in \mathcal{K}$ such that $(N^*N\xi | \eta) = 0$ for all $\xi \in \mathcal{K}$, then $(\xi | N^*N\eta) = 0$ for all $\xi \in \mathcal{K}$. So $N^*N\eta = 0$. So $\eta = 0$ since $\eta \in \mathcal{L}^{\perp}$. Similarly, the ranges of N and of N^* are dense in $\mathcal{K}$. We define $U(\sqrt{(N^*N)}\,\xi) = N\xi$ for all $\xi \in \mathcal{K}$. Since the range of N^*N is dense in $\mathcal{K}$, so is the range of $\sqrt{(N^*N)}$. $\| \sqrt{(N^*N)}\,\xi \|^2 = (\sqrt{(N^*N)}\xi | \sqrt{(N^*N)}\xi)$ $= (N^*N\xi | \xi) = (N\xi | N\xi) = \| N\xi \|^2$. So U is isometric and well-defined. For c in the complex numbers, we have $U(c\sqrt{(N^*N)}\,\xi) = U(\sqrt{(N^*N)}\,c\xi) = N(c\xi) = \bar{c}N\xi$. So U is conjugate homogenous. Since the range of N is dense in $\mathcal{K}$, we can extend U to an anti-unitary operator by conjugate linearity and continuity.

We now need to verify that U commutes with N and with N^*. We have that N and N^* commute with $\sqrt{(N^*N)}$. $(UN)\sqrt{(N^*N)}\,\xi = U\sqrt{(N^*N)}(N\xi) = N(N\xi) = N(U\sqrt{(N^*N)}\xi) = (UN)\sqrt{(N^*N)}\,\xi$. So $UN = NU$ on a dense linear manifold of $\mathcal{K}$ and hence on all of $\mathcal{K}$. Similarly, $UN^* = N^*U$. |

The above construction is analogous to the usual polar decomposition for operators; p. 284, [3].

<u>Lemma 2.7</u> Let U be an anti-unitary operator on $\mathscr{H}$. Let $\mathcal{A}$ be a weakly closed commutative symmetric ring with identity satisfying $U\mathcal{A}U^* = \mathcal{A}$ and $U^2 \in \mathcal{A}$. Let $\mathcal{P}$ denote the set of projection operators of $\mathcal{A}$. Then $\mathcal{P} = \mathcal{P}_0 \oplus \mathcal{P}_1 \oplus \mathcal{P}_2$, where

1) $P \in \mathcal{P}_0$ implies that $UPU^* = P$
2) $\mathcal{P}_2 = U\mathcal{P}_1U^*$ or $\mathcal{P}_2 = \{UP_1U^* | P_1 \in \mathcal{P}_1\}$
3) $\mathcal{P}_0$, $\mathcal{P}_1$, $\mathcal{P}_2$ all have maximum elements in the order on projections such that $T_1 \leq T_2$ is equivalent to $T_1T_2 = T_2T_1 = T_1$. Moreover, each $\mathcal{P}_i$ is the set of all projections in $\mathcal{A}$ less than or equal to the maximum projection in $\mathcal{P}_i$, for $0 \leq i \leq 2$.

Note: This lemma is so algebraically simple that it can probably be found proven more generally elsewhere.

Proof: Let $\pi(P) = UPU^*$. Since $U\mathcal{A}U^* = \mathcal{A}$, we have $\pi(\mathcal{P}) \subset \mathcal{P}$. $\pi^2(P) = \pi(UPU^*) = U^2PU^{*2} = PU^2U^{*2} = P$ since $U^2 \in \mathcal{A}$.

We consider subsets $\tilde{\mathcal{P}}$ of $\mathcal{P}$ satisfying $P_1, P_2 \in \tilde{\mathcal{P}}$ implies $\pi(P_1) \perp P_2$. The zero projection forms such a set. The union of an increasing tower of such sets is again such a set. By the Hausdorff Maximality Principle (H.M.P.), there is a maximal set $\mathcal{P}_1$ in the class of all $\tilde{\mathcal{P}}$. The supremum of an increasing tower of projections in $\mathcal{P}_1$ is again a projection in $\mathcal{P}_1$, since $\mathcal{A}$ is weakly closed. Let Q_1 be a maximal projection in $\mathcal{P}_1$, again using H.M.P. . We claim that Q_1 is the maximum or largest element in $\mathcal{P}_1$. Let $P \in \mathcal{P}_1$. $P-Q_1P$ is in $\mathcal{P}_1$ since $(P-Q_1P) \leq P$ and $P \in \mathcal{P}_1$, $P' \in \mathcal{P}$, $P' \leq P$ imply $P' \in \mathcal{P}_1$. But $Q_1 \oplus (P-Q_1P) \geq Q_1$, and $Q_1 \oplus (P-Q_1P)$ is in $\mathcal{P}_1$. So

$P-Q_1P = 0$ or $P \leq Q_1$. Thus Q_1 is the maximum or largest element of $\mathcal{P}_1$. π preserves products, direct sums, and hence ordering on $\mathcal{P}$. $P \in \mathcal{P}$, $P \leq Q_1$ imply that $\pi(P) \leq \pi(Q_1)$; $R \in \mathcal{P}$, $R \perp \pi(Q_1)$ imply that $R \perp \pi(P)$. So $P \in \mathcal{P}$, $P \leq Q_1$ imply that $P \in \mathcal{P}_1$.

$\pi(Q_1)$ is the maximum element of $\pi(\mathcal{P}_1)$. We set $\pi(\mathcal{P}_1) = \mathcal{P}_2$; we set $\pi(Q_1) = Q_2$. If $P \in \mathcal{P}$, $P \leq Q_2 = \pi(Q_1)$, then $\pi(P) \leq \pi^2(Q_1) = Q_1$. So $\pi(P) \in \mathcal{P}_1$ and $\pi^2(P) = P \in \pi(\mathcal{P}_1) = \mathcal{P}_2$.

Now we let $1-(Q_1 \oplus Q_2) = Q_o$. We suppose that $P \in \mathcal{P}$, $P \leq Q_o$. $P \perp Q_1$, $P \perp Q_2$ imply that $\pi(P) \perp Q_1$, $\pi(P) \perp Q_2$ since $\pi(Q_1) = Q_2$ and $\pi(Q_2) = \pi^2(Q_1) = Q_1$. In order to show that $\pi(P) = P$ for $P \leq Q_o$, $P \in \mathcal{P}$, we consider $\pi(P)-\pi(P)P$.

$\pi[\pi(P)-\pi(P)P] = P-\pi(P)P$. But $[P-\pi(P)P][\pi(P)-\pi(P)P] =$
$P\pi(P)-\pi(P)P-\pi(P)P+[\pi(P)P]^2 = 0$

So $\pi[\pi(P)-\pi(P)P] \perp [\pi(P)-\pi(P)P]$. Also, $\pi(P)-\pi(P)P$ and $P-\pi(P)P$ are orthogonal to Q_1 and Q_2 since both P and $\pi(P)$ are. By the maximality of $\mathcal{P}_1$ and of Q_1, we have that $\pi(P)-\pi(P)P = 0 = P-\pi(P)P$. So $P = \pi(P)$. In short, $P \in \mathcal{P}$, $P \leq Q_o$ imply that $\pi(P) = P$. Thus we let $\mathcal{P}_o = \{P \in \mathcal{P} | P \leq Q_o\}$.

If $P \in \mathcal{P}$, then $P = Q_oP \oplus Q_1P \oplus Q_2P$; $Q_oP \in \mathcal{P}_o$, $Q_1P \in \mathcal{P}_1$, $Q_2P \in \mathcal{P}_2$. |

We are now in a position to prove a representation theorem for normal conjugate operators. As is often the case, the examples, such as 2.51 and 2.52, are analyzed first and the techniques of proof in the general case stumbled upon later.

Theorem 2.8 Let N be a normal conjugate operator on $\mathcal{H}$. Then there is a maximal commutative symmetric ring $\mathcal{Q}$ such that on M, the maximal ideal space of $\mathcal{Q}$, we have $(N\xi)(m) = g(m)\overline{\xi(T(m))}$ for $\xi(m) \in L^2(M,\mu)$ isomorphic to $\mathcal{H}$, where T is a measure-preserving transformation of M with respect to the measure μ determined by some cyclic vector ξ_0. Also, we have $|g(m)| = |g(T(m))|$ almost everywhere with respect to μ. Moreover, the transformation T is such that T^2 is the identity map on M; g(m) is a bounded measurable function. Every such mapping of $\mathcal{H}$ into itself is a normal conjugate operator.

Proof: We assume $\| N \| < 1$. Let $\mathcal{L} = \eta_N$. By lemma 2.6, we have $\mathcal{L} = \eta_N = \eta_{N*} = \eta_{N*N} = \eta_{NN*}$. Let $\mathcal{K} = \mathcal{H} \ominus \mathcal{L}$. All further remarks pertain to the restriction of N to $\mathcal{K}$. Let $N = U\sqrt{(N^*N)}$ as in lemma 2.6 . U commutes with N and N*. Using the spectral representation theorem for the normal operator N^2, we write $N^2 = \int_{\mathcal{C}} \lambda \, dP_\lambda$, where $\mathcal{C}$ is the complex plane. Let $\mathcal{S}$ be the subset of the complex plane excluding the real axis; $\mathcal{S} = \{x+iy \mid y \neq 0\}$. Let $\mathcal{K}_1 = (\int_{\mathcal{S}} 1 dP_\lambda)\mathcal{K}$. Let $\mathcal{K}_3 = (\int_{\mathbb{R}} 1 dP_\lambda)\mathcal{K}$, where $\mathbb{R}$ is the real line. $\mathcal{K}_1$ and $\mathcal{K}_3$ reduce N^2, and $\mathcal{K} = \mathcal{K}_1 \oplus \mathcal{K}_3$ by the spectral theorem. We claim that $\mathcal{K}_1$ and $\mathcal{K}_3$ each reduce the conjugate operator N. $N^2 = \int_{\mathcal{C}} \lambda \, dP_\lambda = UN^2U^* = \int_{\mathcal{C}} \bar{\lambda} \, d(UP_\lambda U^*)$. By the uniqueness of spectral measure, cf. p. 71, [1], we have that $P(\bar{\Delta}) = UP(\Delta)U^*$ for Δ a Borel subset of $\mathcal{C}$. If Δ' is a subset of $\mathbb{R}$, then we have that $UP(\Delta')U^* = P(\Delta')$. Let $Q_1 = \int_{\mathrm{Im}\,\lambda<0} 1 dP_\lambda = P(\Delta_1)$, where $\Delta_1 = \{\lambda \in \mathcal{C} \mid \mathrm{Im}\,\lambda < 0\}$. Let $Q_2 =$

$\int_{\text{Im }\lambda>0} 1dP_\lambda = P(\Delta_2)$, where $\Delta_2 = \{\lambda\in\mathbb{C} | \text{Im }\lambda > 0\}$. Thus $UQ_1U^* = Q_2$. Also, $\mathcal{K}_1 = Q_1(\mathcal{K}_1) \oplus Q_2(\mathcal{K}_1)$. Our plan is to deal with $\mathcal{K}_1$ and $\mathcal{K}_3$ separately but analogously.

First, we consider $\mathcal{K}_1$. $\mathcal{K}_1 = Q_1(\mathcal{K}_1) \oplus Q_2(\mathcal{K}_1)$, where each summand is a reducing space for N^2. Since U commutes with N and N*, it follows that U^2 commutes with N^2 and $N^{*2} = (N^2)^*$; $Q_1(\mathcal{K}_1)$ and $Q_2(\mathcal{K}_1)$ both reduce U^2 since they are both spectral subspaces of N^2. Let $\mathcal{B}_1$ be the weakly closed symmetric commutative ring generated by N^2 and U^2. $\sqrt{(N^*N)} = \sqrt[4]{(N^{*2}N^2)} \in \mathcal{B}_1$. Since linear combinations of products of U^2, U^{*2}, N^2, and N^{*2} are weakly dense in $\mathcal{B}_1$, we have $U\mathcal{B}_1U^* = \mathcal{B}_1$. Now we will show that we can extend $\mathcal{B}_1$ to a maximal commutative symmetric overring $\mathcal{A}_1$ on $\mathcal{K}_1$ satisfying $U\mathcal{A}_1U^* = \mathcal{A}_1$.

Since Q_1 commutes with both U^2 and N^2, we have that $\mathcal{B}_1[Q_1(\mathcal{K}_1)] \subset Q_1(\mathcal{K}_1)$. We pick ζ_1 of norm one in $Q_1(\mathcal{K}_1)$. If $\{\overline{\mathcal{B}_1\zeta_1}\} \neq Q_1(\mathcal{K}_1)$, then we pick ζ_2 of norm one in the subspace $\{Q_1(\mathcal{K}_1)\} \ominus \{\overline{\mathcal{B}_1\zeta_1}\}$. Using the symmetry of the ring $\mathcal{B}_1$, it is easy to see that $\{\overline{\mathcal{B}_1\zeta_1}\} \perp \{\overline{\mathcal{B}_1\zeta_2}\}$. Proceeding in this fashion and employing the Hausdorff Maximality Principle, we can write $Q_1(\mathcal{K}_1) = \sum_{i=1}^{\infty}\oplus\{\overline{\mathcal{B}_1\zeta_i}\}$ where $\zeta_i \in Q_1(\mathcal{K}_1)$ for $1 \leq i < \infty$. Since $UQ_1U^* = Q_2$, we have $UQ_1U^*(\mathcal{K}_1) = Q_2(\mathcal{K}_1)$. Since $U^*(\mathcal{K}_1) = \mathcal{K}_1$, it is true that $UQ_1(\mathcal{K}_1) = Q_2(\mathcal{K}_1)$. Let R_i be the projection operator with range $\{\overline{\mathcal{B}_1\zeta_i}\}$ for $1 \leq i < \infty$. Then R_i commutes with $\mathcal{B}_1$ since $R_i(\mathcal{K}_1)$ reduces $\mathcal{B}_1$, for $1 \leq i < \infty$. We claim that UR_iU^* is the projection onto $\{\overline{\mathcal{B}_1U\zeta_i}\}$ for $1 \leq i < \infty$. Let $B \in \mathcal{B}_1$. Let $B = UB'U^*$, where $B' \in \mathcal{B}_1$. Then we have:

$$UR_iU^*BU\zeta_j = UR_iU^*UB'U^*U\zeta_j = UR_iB'\zeta_j = \begin{cases} 0 \text{ if } i \neq j \\ UR_iB'\zeta_i \text{ if } i = j \end{cases}$$

But $UR_iB'\zeta_i = UB'\zeta_i = U^2BU^*\zeta_i = BU\zeta_i$. So UR_iU^* is the projection onto $\overline{\{\mathcal{B}_1U\zeta_i\}}$ for $1 \leq i < \infty$, and $Q_2(\mathcal{K}_1) = \sum_{i=1}^{\infty}\oplus\overline{\{\mathcal{B}_1U\zeta_i\}}$. We adjoin $\{R_i\}_{i=1}^{\infty}$ and $\{UR_iU^*\}_{i=1}^{\infty}$ to the restriction of the ring $\mathcal{B}_1$ to $\mathcal{K}_1$. The weak closure of this new ring is called $\mathcal{A}_1$. Clearly $U\mathcal{A}_1U^* = \mathcal{A}_1$. $\mathcal{A}_1$ has $(\sum_{i=1}^{\infty}2^{-i}\zeta_i + \sum_{i=1}^{\infty}2^{-i}U\zeta_i)$ as a cyclic vector. So $\mathcal{A}_1$ is a maximal commutative symmetric ring on $\mathcal{K}_1$. The restriction of $\sqrt{(N^*N)}$ to $\mathcal{K}_1$ is in $\mathcal{A}_1$; the restriction of U^2 to $\mathcal{K}_1$ is in $\mathcal{A}_1$ by definition of $\mathcal{B}_1$.

Now we consider $\mathcal{K}_3 = (\int_{\mathbb{R}} 1dP_\lambda)\mathcal{K}$. $\mathcal{K}_3$ reduces U and N^2 since $\mathcal{K}_1$ does. Hence $\mathcal{K}_3$ reduces $N = U\sqrt{(N^*N)}$. The restriction of N^2 to $\mathcal{K}_3$ is Hermitian, since it has real spectrum. The following remarks apply to $\mathcal{K}_3$ and restrictions of operators or of conjugate operators to $\mathcal{K}_3$.

$N = U\sqrt{(N^*N)}$. $N^2 = U^2N^*N$. $N^* = \sqrt{(N^*N)}U^* = U^*\sqrt{(N^*N)}$. $N^{*2} = U^{*2}N^*N$. $N^*N\mathcal{K}_3$ is dense in $\mathcal{K}_3$, since $N^*N\mathcal{K}$ is dense in $\mathcal{K}$ and $\mathcal{K}_3$ reduces N^*N. Thus $N^2 = N^{*2} = U^2(N^*N) = U^{*2}(N^*N)$ implies that $U^2 = U^{*2}$. So U^2 is Hermitian on $\mathcal{K}_3$. Let $U^2 = P_1 - P_2$, where $P_1 \oplus P_2$ is the projection onto $\mathcal{K}_3$. $U^4 - U^2 = P_2$. $U^4 - P_2 = P_1$. So P_1 and P_2 commute with N and with U. $P_1(\mathcal{K}_3)$ and $P_2(\mathcal{K}_3)$ reduce N and U. $U^2 = 1$ on $P_1(\mathcal{K}_3)$ and hence $U = U^*$ on $P_1(\mathcal{K}_3)$. Thus $N = N^*$ on $P_1(\mathcal{K}_3)$. $U^2 = -1$ or $U = -U^*$ on $P_2(\mathcal{K}_3)$. So $N = -N^*$ on $P_2(\mathcal{K}_3)$.

On $P_1(\mathcal{K}_3)$, we have $N = N^*$. By theorem 2.1, there is a maximal commutative symmetric ring $\mathcal{Q}_2$ containing N^2 on $P_1(\mathcal{K}_3)$. For $H \in \mathcal{Q}_2$, H Hermitian, we have $HU = UH$ or $NH = HN$ on $P_1(\mathcal{K}_3)$.

On $P_2(\mathcal{K}_3)$ the situation is slightly more difficult. We note that $N = -N^*$ on $P_2(\mathcal{K}_3)$. For $\xi \in P_2(\mathcal{K}_3)$, $(N\xi|\xi) = (N^*\xi|\xi) = -(N\xi|\xi)$. So $(N\xi|\xi) = 0$. $(N^{2n}\xi|N\xi) = (N^nN^n\xi|N\xi) = (N^{*n}N\xi|N^n\xi) = [-1]^n(N^nN\xi|N^n\xi) = [-1]^n(NN^n\xi|N^n\xi) = 0$. So if $P(x)$ is a polynomial, we have $(P(N^2)\xi|N\xi) = 0$. We recall that $\mathcal{B}_1$ is the weakly closed symmetric commutative ring generated by U^2 and N^2. We have $\overline{\{\mathcal{B}_1\xi\}} \perp N\xi$ for $\xi \in P_2(\mathcal{K}_3)$. Now $\sqrt{(N^*N)} \in \mathcal{B}_1$, and $N = \sqrt{(N^*N)}\,U$. So $\overline{\{\mathcal{B}_1\xi\}} \perp U\xi$. Now we are in a position to construct an overring of the ring $\mathcal{B}_1$ restricted to $P_2(\mathcal{K}_3)$ as we did with $\mathcal{B}_1$ restricted to $\mathcal{K}_1$.

We fix ξ_1 of norm one in $P_2(\mathcal{K}_3)$. Let S_1 be the projection onto $\overline{\{\mathcal{B}_1\xi_1\}}$. Since $\mathcal{B}_1$ is symmetric and $U\xi_1 \perp \overline{\{\mathcal{B}_1\xi_1\}}$, we have $\overline{\{\mathcal{B}_1U\xi_1\}} \perp \overline{\{\mathcal{B}_1\xi_1\}}$. We claim US_1U^* is the projection onto $\overline{\{\mathcal{B}_1U\xi_1\}}$. $\overline{\{\mathcal{B}_1U\xi_1\}} = \overline{\{U\mathcal{B}_1\xi_1\}} = U\overline{\{\mathcal{B}_1\xi_1\}}$. If $\xi \perp \overline{\{\mathcal{B}_1U\xi_1\}}$, then $\xi \perp U\overline{\{\mathcal{B}_1\xi_1\}}$ or $U^*\xi \perp \overline{\{\mathcal{B}_1\xi_1\}}$. So $S_1U^*\xi = 0 = US_1U^*\xi$. If, on the other hand, $\xi \in \overline{\{\mathcal{B}_1U\xi_1\}} = U\overline{\{\mathcal{B}_1\xi_1\}}$, then $U^*\xi \in \overline{\{\mathcal{B}_1\xi_1\}}$. $S_1U^*\xi = U^*\xi$. So $US_1U^*\xi = \xi$.

Now we choose ξ_2 of norm one in $[\overline{\{\mathcal{B}_1\xi_1\}} \oplus \overline{\{\mathcal{B}_1U\xi_1\}}]^{\perp}$, where we take the orthogonal complement in $P_2(\mathcal{K}_3)$. Since $\overline{\{\mathcal{B}_1\xi_1\}} \oplus \overline{\{\mathcal{B}_1U\xi_1\}}$ reduces the ring $\mathcal{B}_1$, N, and U, we can repeat the above argument to obtain $\overline{\{\mathcal{B}_1\xi_2\}} \perp \overline{\{\mathcal{B}_1U\xi_2\}}$ and S_2, US_2U^* as the projections onto these subspaces. By a maximality argument, we can write $P_2(\mathcal{K}_3) = \sum_{i=1}^{\infty}\oplus\overline{\{\mathcal{B}_1\xi_i\}} \oplus \sum_{i=1}^{\infty}\oplus\overline{\{\mathcal{B}_1U\xi_i\}}$ where the norm of ξ_i is one for $1 \leq i < \infty$. We let S_i be the projection onto $\overline{\{\mathcal{B}_1 \xi_i\}}$ for $1 \leq i < \infty$. Then US_iU^* is the projection onto $\overline{\{\mathcal{B}_1 U\xi_i\}}$ for $1 \leq i < \infty$. We adjoin $\{S_i\}_{i=1}^{\infty}$ and $\{US_iU^*\}_{i=1}^{\infty}$ to the restriction of the ring $\mathcal{B}_1$ to $P_2(\mathcal{K}_3)$. We denote the weak closure of this symmetric commutative overring by $\mathcal{A}_3$. Then $\sum_{i=1}^{\infty} 2^{-i}\xi_i + \sum_{i=1}^{\infty} 2^{-i}U\xi_i$ is a cyclic vector for $\mathcal{A}_3$. So $\mathcal{A}_3$ is a maximal commutative symmetric ring on $P_2(\mathcal{K}_3)$. $U\mathcal{A}_3U^* = \mathcal{A}_3$ by construction.

We have $\mathcal{A}_1$ on $\mathcal{K}_1$, $\mathcal{A}_2$ on $P_1(\mathcal{K}_3)$, and $\mathcal{A}_3$ on $P_2(\mathcal{K}_3)$, each ring maximal commutative symmetric on its subspace. Let $\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_2 \oplus \mathcal{A}_3$. $\mathcal{A}$ is a maximal commutative symmetric ring on $\mathcal{K}$. Since $\mathcal{K}_1$, $P_1(\mathcal{K}_3)$, and $P_2(\mathcal{K}_3)$ each reduce U, and $U\mathcal{A}_iU^* = \mathcal{A}_i$ for $1 \leq i \leq 3$, we have $U\mathcal{A}U^* = \mathcal{A}$. Also, U^2 and $\sqrt{(N^*N)}$ are both in $\mathcal{A}$.

We are now ready to apply lemma 2.7 to $\mathcal{A}$ and $\mathcal{P}$, the set of projection operators of $\mathcal{A}$. We can write $\mathcal{P} = \mathcal{P}_o \oplus \mathcal{P}_1 \oplus \mathcal{P}_2$ where $P \in \mathcal{P}_o$ implies $UPU^* = P$, $P \in \mathcal{P}_1$ implies $UPU^* \in \mathcal{P}_2$, and $P \in \mathcal{P}_2$ implies $UPU^* \in \mathcal{P}_1$. Let S_o, S_1, and S_2 be the maximum projections of $\mathcal{P}_o$, $\mathcal{P}_1$, and $\mathcal{P}_2$ respectively. We choose θ_o of norm $[\sqrt{3}]^{-1}$ and cyclic for $S_o\mathcal{A}S_o$ on $S_o\mathcal{K}$. We choose θ_1 of norm $[\sqrt{3}]^{-1}$ and cyclic for $S_1\mathcal{A}S_1$ on $S_1\mathcal{K}$. Now $\| U\theta_1 \| = [\sqrt{3}]^{-1}$. Since $US_1U^* = S_2$, we have $US_1\mathcal{K} = S_2\mathcal{K}$ and $US_1 = S_2U$. We find:

$$S_2\mathcal{A}S_2U\theta_1 = S_2\mathcal{A}US_1\theta_1 = S_2\mathcal{A}U\theta_1 = S_2(U\mathcal{A}U^*)U\theta_1 = S_2U\mathcal{A}\theta_1 = S_2US_1\mathcal{K} = S_2\mathcal{K}.$$

So $U\theta_1$ is cyclic for $S_2\mathcal{A}S_2$ on $S_2\mathcal{K}$. For $P \in \mathcal{P}_1$, it holds that $(P\theta_1|\theta_1) = (UP\theta_1|U\theta_1) = (UPU^*(U\theta_1)|U\theta_1)$. Let $\xi_o = \theta_o \oplus \theta_1 \oplus U\theta_1$. Let M be the maximal ideal space of $\mathcal{A}$, and let μ be the regular

Borel measure corresponding to the cyclic vector ξ_o. We let T denote the map of clopen sets of M which corresponds to the mapping of P to UPU^* for $P \in \mathcal{P}$; if $\hat{P}(m) = \chi_\Delta(m)$ and $\widehat{(UPU^*)}(m) = \chi_{\Delta'}(m)$ where Δ and Δ' are clopen sets in M, we set $T(\Delta) = \Delta'$. By construction of the cyclic vector ξ_o, namely, since $(P\theta_1|\theta_1) = (UPU^*(U\theta_1)|U\theta_1)$ for $P \in \mathcal{P}$, we have that $\mu(\Delta) = \mu(T[\Delta])$ for $\Delta \subset \overline{M}$, Δ clopen. Since the clopen sets of $\overline{M}$ form a complete set of representatives for the measure algebra of (M,μ), we see that T induces an automorphism of the measure algebra. Moreover, we can define T in a pointwise fashion in the following way. If $m \in M$, then $UmU^* \in M$. If m is a maximal ideal and $P \in \mathcal{P}$, then $P \in m$ if and only if $UPU^* \in UmU^*$. Or $\hat{P}(m) = 0$ if and only if $\widehat{(UPU^*)}(UmU^*) = 0$. So if we define $T(m) = UmU^*$, then the pointwise map agrees with the set map T; and hence the same notation is justified.

We have $\xi_o(m) = 1$ for all $m \in M$. Let $f(m) = (U\xi_o)(m)$. Now $UP = (UPU^*)U$, so $U(\chi_\Delta) = \chi_{T(\Delta)}U$ for Δ clopen, or $U(\chi_\Delta)(m)\xi_o(m) = (\chi_{T(\Delta)})(m)(U\xi_o)(m)$. Since U is anti-unitary, we have that $\| U(\chi_\Delta)(m) \|^2 = \mu(\Delta) = \int_{T(\Delta)} |f(m)|^2 d\mu(m) = \mu(T[\Delta])$. So $|f(m)| = 1$ almost everywhere. If $\sum_{i=1}^{n} a_i \chi_{\Delta_i} \in L^2(M,\mu)$, we have:

$$U\left(\sum_{i=1}^{n} a_i \chi_{\Delta_i}\right) = \sum_{i=1}^{n} \overline{a}_i U(\chi_{\Delta_i}) = \sum_{i=1}^{n} \overline{a}_i \; f|_{T(\Delta_i)}(m) = f(m) \sum_{i=1}^{n} \overline{a}_i \chi_{T(\Delta_i)}.$$

By continuity and linearity, we have $(U\xi)(m) = f(m)\xi(T[m])$ for $\xi(m) \in L^2(M,\mu)$. Now $N = U\sqrt{(N^*N)} = \sqrt{(N^*N)}\,U$. So $(N\xi)(m) =$

$(\widehat{\sqrt{(N^*N)}})(m)\ f(m)\ \overline{\xi(T[m])}$. Let $g(m) = (\widehat{\sqrt{(N^*N)}})(m)\ f(m)$. Then $(N\xi)(m) = g(m)\overline{\xi(T[m])}$. $(N\xi|\eta) = \int_M g(m)\ \overline{\xi(T[m])}\,\overline{\eta(m)}\,d\mu(m) = \int_M g(T[m])\ \overline{\xi(m)}\,\overline{\eta(T[m])}\,d\mu(m) = (N^*\eta|\xi)$ for all $\xi,\eta \in \mathcal{K}$. Thus $(N^*\xi)(m) = g(T[m])\overline{\xi(T[m])}$.

Since $N^*N\xi = NN^*\xi$, we have $g(T[m])\overline{g(T[m])}\xi(m) = g(m)\overline{g(m)}\xi(m)$ So $|g(T[m])| = |g(m)|$ almost everywhere with respect to μ.

For completeness' sake, we could adjoin to $\mathcal{A}$ any maximal commutative symmetric ring on $\mathcal{L} = \mathcal{N}_N$, defining f to be the zero function and T to be the identity map on the maximal ideal space of the adjoined ring.|

CHAPTER III

THE INVARIANT SUBSPACE PROBLEM

We now consider the well known question of whether each bounded operator on a countably infinite dimensional Hilbert space has a non-trivial invariant subspace. We suppose that the answer is no and attempt to find candidates for an example of an operator without a non-trivial invariant subspace.

Let S be an operator on $\mathscr{H}$ without a non-trivial invariant subspace. Let $S = UA$, where UA is the polar decomposition of S. We claim that in this case U is unitary. Since $\mathcal{N}_S$ is invariant under S, we have $\mathcal{N}_S = \{0\}$. Since $\mathcal{N}_A = \mathcal{N}_{\sqrt{(S^*S)}} \subset \mathcal{N}_S$, we have $\mathcal{N}_A = \{0\}$. Since $\mathcal{D}_U = \overline{\mathcal{R}}_A$, we have $\mathcal{D}_U = \mathscr{H}$. Since $\mathcal{R}_S$ is invariant under S, it holds that $\overline{\mathcal{R}}_S = \mathscr{H} = \mathcal{R}_U$. Thus U is a partial isometry with domain space $\mathscr{H}$ and range space $\mathscr{H}$. Hence U is unitary.

If S has no non-trivial invariant subspace, then U and A have no common non-trivial invariant subspace. In this chapter we give four examples of operators $\{S_i\}_{i=1}^{4}$ having dense range and zero null-space such that S_i has polar decomposition U_iA_i where U_i and A_i have no common non-trivial invariant subspace. At least one of the examples to be given here, the first, is well known.

To construct these examples, we note the following facts. If A is a Hermitian operator on $\mathscr{H}$ such that A generates a maximal commutative symmetric ring $\mathcal{A}(A)$ in the weak operator topology, then every invariant subspace for A corresponds to a projection in $\mathcal{A}(A)$. For if $\mathcal{K}$ is invariant under A and hence under $A^* = A$, then $P_{\mathcal{K}}A = AP_{\mathcal{K}}$, where $P_{\mathcal{K}}$ is the projection onto $\mathcal{K}$. Thus $P_{\mathcal{K}}$ is in $\mathcal{A}(A)$ inasmuch as $\mathcal{A}(A)$ is generated by A.

If U is a unitary operator such that there is a sequence $\{n_p\}_{p=1}^{\infty}$ of positive integers for which U^{n_p} converges to U^* in the strong topology and such that U and U^* generate a maximal commutative symmetric ring $\mathcal{A}(U)$ in the weak topology, then again every subspace $\mathcal{L}$ invariant under U corresponds to a projection in $\mathcal{A}(U)$. The same conclusion holds if we replace U^{n_p} by a net of polynomials in U or by a net of continuous or Borel functions of U which converges strongly to U^*.

Thus some operators have an easily described set of invariant subspaces. We are led to the following definition.

Definition 3.0 An operator W on $\mathcal{H}$ is said to be completely normal if and only if each invariant subspace for W is a reducing subspace for W.

Clearly an operator W is completely normal if and only if W^* is completely normal. Every Hermitian operator is completely normal. We have no characterization of which normal operators are completely normal. We believe that there exist non-normal completely normal operators, but we have no examples known to be such. An operator without a non-trivial invariant subspace would be an example.

Lemma 3.1 A unitary operator U is completely normal if and only if for each subspace $\mathcal{K}$ such that $U\mathcal{K} \subset \mathcal{K}$, it is true that $U\mathcal{K} = \mathcal{K}$.

Proof: We suppose that U is completely normal; let $\mathcal{K}$ be such that $U\mathcal{K} \subset \mathcal{K}$. If $\xi \in \mathcal{K} \ominus U\mathcal{K}$, then $\xi \perp U\mathcal{K}$ implies that $U^*\xi \perp \mathcal{K}$. Thus $\xi = 0$ or $U\mathcal{K} = \mathcal{K}$.

Now we suppose that U is such that $U\mathcal{K} \subset \mathcal{K}$ implies that $U\mathcal{K} = \mathcal{K}$. Let $\mathcal{K}_1$ be invariant under U. Then $U\mathcal{K}_1 = \mathcal{K}_1, U^*U\mathcal{K}_1 = U^*\mathcal{K}_1$, and $\mathcal{K}_1 = U^*\mathcal{K}_1$. So $\mathcal{K}_1$ reduces U, and U is completely normal.|

<u>Lemma 3.2</u> If N is a normal, completely normal operator such that N and N* generate a maximal commutative symmetric ring $\mathcal{A}(N)$ in the weak topology, then the projection onto each invariant subspace of N is an element of $\mathcal{A}(N)$.

Proof: Let $N\mathcal{K} \subset \mathcal{K}$; let P be the projection onto $\mathcal{K}$. $N\mathcal{K} \subset \mathcal{K}$ implies that PNP = NP. Since N is completely normal, it holds that NP = PN, N*P = PN*, and $P \in \mathcal{A}(N)$ by the maximality of $\mathcal{A}(N)$.|

We now define some completely normal unitary operators which will be used in constructing the candidates for an example of an operator without an invariant subspace.

Consider the measure space $\{[0,2\pi],\mu\}$, where μ is Lebesgue measure. If $\theta \in \mathbb{R}$, let $\tilde{\theta}$ be such that $0 \leq \tilde{\theta} < 2\pi$, and $\tilde{\theta}$ is congruent to θ modulo 2π. For $\theta \in [0,2\pi]$, let $T(\theta) = (\widetilde{\theta-1})$. T is a measure-preserving ergodic transformation; p. 25-30,[2]. $T^{-1}(\theta) = (\widetilde{\theta+1})$. $\{T^n(0)\}_{n=0}^{\infty}$ is dense in $[0,2\pi]$ in the usual interval topology. Translation of functions is continuous in $L^2([0,2\pi])$, $\int_0^{2\pi}|f(\theta)-f(\widetilde{\theta+a})|^2 d\theta$ goes to zero as a goes to zero. In the first example, we will define $U_1 f(\theta) = f(T[\theta])$ for $f \in L^2([0,2\pi])$. U_1 is a unitary operator; $U_1^* f(\theta) = f(T^{-1}[\theta])$. Since $\{T^n(0)\}_{n=0}^{\infty}$ is dense in $[0,2\pi]$, we can find a sequence of integers $\{k_p\}_{p=1}^{\infty}$ such that $|T^{k_p}(0) - T^{-1}(0)|$ goes to zero as p goes to infinity. Thus $U_1^{k_p}$ converges strongly to U_1^*, and U_1 is completely normal.

Our final completely normal unitary operator depends on analytic function theory rather than ergodic theory. We claim that for α such that $0 < \alpha < 2\pi$, the span of $\{e^{in\theta}\}_{n=1}^{\infty}$ is dense in the continuous functions on $[0,\alpha]$ under the supremum norm. The proof depends on theorems 3.9 and 3.6 and on corollary 3.3.1 of [6]. If the span of $\{e^{in\theta}\}_{n=1}^{\infty}$ is not dense in the continuous functions on $[0,\alpha]$, then there is a regular Borel measure given by φ, a normalized function of bounded variation on $[0,\alpha]$ such that $\int_0^{\alpha} e^{in\theta} d\varphi = 0$ for $n \geq 1$. Let $k = \varphi(0^+) - \varphi(0)$. We define ψ as follows:

$$\psi(\theta) = \begin{cases} \varphi(\theta) - 2^{-1}k \text{ for } 0 < \theta < \alpha \\ 2^{-1}[\varphi(\alpha) - \psi(\alpha^-)] - 2^{-1}k \text{ for } \theta = \alpha \\ \varphi(\alpha) - 2^{-1}k \text{ for } \alpha < \theta < 2\pi \\ \varphi(0) \text{ for } \theta = 0, \text{ and } \varphi(\alpha) \text{ for } \theta = 2\pi \end{cases}$$

Thus ψ is a normalized function of bounded variation such that $\int_0^{2\pi} e^{in\theta} d\psi = 0$ for $n \geq 1$. By theorem 3.9 of [6], ψ is absolutely continuous and $\psi'(\theta) = \lim_{r\to 1} g(re^{in\theta})$ almost everywhere, with g analytic and beschränktartige by theorem 3.6 .

$\psi'(\theta) = 0$ for $\alpha < \theta < 2\pi$. So by corollary 3.3.1 of [6], $g \equiv 0$. Thus $\psi \equiv 0$, and $\varphi \equiv 0$. So $\{e^{in\theta}\}_{n=1}^{\infty}$ are dense in the continuous functions on $[0,\alpha]$ for $0 < \alpha < 2\pi$.

Hence the span of $\{e^{i\alpha n\theta}\}_{n=1}^{\infty}$ is dense in the continuous on $[0,2\pi]$ in the supremum norm for $0 < \alpha < 2\pi$. We fix α between 0 and 2π. For $f \in L^2([0,2\pi])$, we define $U_3 f(\theta) = e^{i\alpha\theta} f(\theta)$. U_3 is unitary. Since polynomials in $e^{i\alpha\theta}$ can be found to approximate $e^{-i\alpha\theta}$ in the supremum norm, we know that these polynomials in U_3 approximate U_3^* in the operator norm. So U_3 is completely normal.

Example 1 As we said earlier, we define $U_1 f(\theta) = f(\theta \tilde{-} 1) = f(T[\theta])$ for $f \in L^2([0,2\pi])$. Let $A_1 f(\theta) = \theta f(\theta)$. By the Weierstrass approximation theorem, the norm closure of the ring generated by A_1 contains all operators given by multiplications by continuous functions which vanish at zero. Hence the weakly closed ring generated by A_1 is $L^\infty([0,2\pi])$, a maximal commutative symmetric ring; p.352, IV,[3]. By lemma 3.2, each invariant subspace of A_1 corresponds to a Borel subset E of $[0,2\pi]$ and is the subspace of square-summable functions which are supported on E. Since T corresponding to U_1 is ergodic on $[0,2\pi]$, the only subsets F of $[0,2\pi]$ such that $T^{-1}(F) = F$ are such that $\mu(F)$ is equal to 0 or to 2π, where μ denotes Lebesgue measure. So U_1 and A_1 have only trivial common invariant subspaces.

We give a second proof of this fact, using that U_1 is completely normal. Since $U_1 e^{in\theta} = e^{inT(\theta)} = e^{in(\theta-1)} = e^{-in}e^{in\theta}$, U_1 has a complete orthonormal set of eigenvectors with discrete eigenvalues. Thus if P_n is the projection onto $\mathcal{S}\{e^{in\theta}\}$ for $-\infty < n < \infty$, we have P_n is in the weakly closed ring generated by U_1 and U_1^*. So by lemma 3.2, each invariant subspace for U_1 is of the form $\mathcal{S}\{e^{in\theta}\}_{n\in J}$, where J is a subset of the integers. Since $|e^{in\theta}| = 1$ for $\theta \in [0,2\pi]$ and $-\infty < n < \infty$, our previous characterization of the invariant subspaces for A_1 shows that U_1 and A_1 have no common non-trivial invariant subspace. We note that A_1 has no eigenvectors and that U_1 has a complete set of eigenvectors. We set $S_1 = U_1 A_1$. We refer the reader to [4] for a more complete discussion of this type of operator.

Example 2 In this example both U_2 and A_2 have a complete set of eigenvectors. Let $\mathcal{H} = \mathcal{S}\{e^{in\theta}\}_{n=0}^{\infty} = H^2([0,2\pi])$. Let $\{\eta_i\}_{i=0}^{\infty}$ be the skewed basis with respect to $\{e^{in\theta}\}_{n=0}^{\infty}$; this construction was given in the beginning of the chapter on partial isometries. We claim that a subspace $\mathcal{K}$ of the form $\mathcal{K} = \mathcal{S}\{e^{in\theta}\}_{n\in A} = \mathcal{S}\{\eta_j\}_{j\in B}$, where A and B are subsets of the integers, is either $\{0\}$ or $\mathcal{H}$. If $\mathcal{K} \neq \{0\}$, we note that $(e^0|\eta_j) \neq 0$ for each $j \in B$; for $(e^0|\eta_j) \neq 0$ for $0 \leq j < \infty$ by construction of the skewed basis. Since $\mathcal{K}$ is of the form $\mathcal{S}\{e^{in\theta}\}_{n\in A}$, we have that $e^0 \in \mathcal{K}$ or $e^0 \perp \mathcal{K}$. If B is non-empty, we have $e^0 \in \mathcal{K}$. Since $\mathcal{K}$ is of the form $\mathcal{S}\{\eta_j\}_{j\in B}$, we have that $\eta_j \in \mathcal{K}$ or $\eta_j \perp \mathcal{K}$ for $0 \leq j < \infty$. Since $e^0 \in \mathcal{K}$ and $(e^0|\eta_j) \neq 0$

for $0 \leq j < \infty$, we have $\eta_j \in \mathcal{K}$. So $\mathcal{K} = \mathcal{H}$.

We let U_2 be the restriction of U_1 to $H^2([0,2\pi]) = \mathcal{J}\{e^{in\theta}\}_{n=0}^{\infty}$, a subspace of $\mathcal{J}\{e^{in\theta}\}_{n=-\infty}^{\infty}$ which reduces U_1. [For an alternate definition of $H^2([0,2\pi])$, see [6].] U_2 is completely normal. Each invariant subspace for U_2 is of the form $\mathcal{J}\{e^{in\theta}\}_{n\in A}$, where A is a subset of the non-negative integers, by lemma 3.2 .

We let $A_2(\sum_{j=0}^{\infty} a_j\eta_j) = \sum_{j=0}^{\infty} a_j[1-2^{-j-1}]\eta_j$. Or, if Q_j is the projection onto $\mathcal{J}\{\eta_j\}$, then $A_2 = \sum_{j=0}^{\infty}[1-2^{-j-1}]Q_j$. Each invariant subspace of A_2 is of the form $\mathcal{J}\{\eta_j\}_{j\in B}$, where B is a subset of the non-negative integers, by lemma 3.2 .

Thus U_2 and A_2 have only the trivial subspaces as common invariant subspaces. We let $S_2 = U_2A_2$. The property of the skewed basis used in this example is the motivation for the choice of the description "skewed."

Example 3 In this example, U_3 has no eigenvectors and A_3 has a complete set of eigenvectors. As before, we fix α such that $0 < \alpha < 2\pi$ and define $U_3f(\theta) = e^{i\alpha\theta}f(\theta)$ for $f \in L^2([0,2\pi])$. If R_n is the projection onto $\mathcal{J}\{e^{in\theta}\}$, we define $A_3 = \sum_{n=-\infty}^{0}(1-3^{n-1})R_n + \sum_{n=1}^{\infty}(1-2^{-n})R_n$. Since each invariant subspace of U_3 corresponds to a Borel subset of $[0,2\pi]$ and each invariant subspace of A_3 is of the form $\mathcal{J}\{e^{in\theta}\}_{n\in D}$, where D is a subset of the integers, we see that U_3 and A_3 have only the trivial subspaces as common invariant subspaces. We set $S_3 = U_3A_3$.

Example 4 In our final example both U_4 and A_4 have no eigenvectors. In fact, the Hilbert space is $L^2([0,2\pi])$, and $U_4 = U_3$. We set $Ve^{in\theta} = e^{i(n+1)\theta}$ or $Vf(\theta) = e^{in\theta}f(\theta)$ for $f \in L^2([0,2\pi])$. We recall that $U_1f(\theta) = f(T[\theta]) = f(\widetilde{\theta-1})$. $U_1Vf(\theta)$ $U_1e^{i\theta}f(\theta) = e^{i(\widetilde{\theta-1})}f(\widetilde{\theta-1}) = e^{-i}e^{i\theta}f(\widetilde{\theta-1}) = e^{-i}VU_1f(\theta)$. So $U_1V = e^{-i}VU_1$. Moreover, any proper non-zero subspace of $L^2([0,2\pi])$ invariant under $U_4 = U_3$ is not invariant under U_1V_1, since T is ergodic.

For $e^{i0\theta} = e^0$, we have that $\{V^je^0\}_{j=-\infty}^{\infty}$ is an orthonormal basis for $L^2([0,2\pi])$. Now $U_1^je^0 = e^0$ for $-\infty < j < \infty$. Thus $U_1V = e^{-i}VU_1$ implies that $(U_1V)^je^0 = c_jV^je^0$, where $|c_j| = 1$ for $-\infty < j < \infty$. So $\{(U_1V)^je^0\}_{j=-\infty}^{\infty}$ is an orthonormal basis for $\mathcal{H} = L^2([0,2\pi])$; and U_1V is the bilateral shift with respect to this basis. So we represent U_1V as a multiplication by $e^{i\theta}$ on $L^2([2\pi,4\pi])$. $U_1Vg(\theta) = e^{i\theta}g(\theta)$ for $g(\theta) \in L^2([2\pi,4\pi])$.

We set $A_4g(\theta) = (\theta-2\pi)g(\theta)$ for $g(\theta) \in L^2([2\pi,4\pi])$. The invariant subspaces for A_4 correspond to Borel subsets of $[2\pi,4\pi]$ by lemma 3.2 . Each such subspace is invariant under U_1V, since U_1V is represented as a multiplication on $L^2([2\pi,4\pi])$. But no non-trivial invariant subspace for U_4 is invariant under U_1V since T is ergodic, as mentioned earlier. So U_4 and A_4 have no non-trivial invariant subspace in common.|

BIBLIOGRAPHY

1. Halmos, P.R. Introduction to Hilbert Space and the Theory of Spectral Multiplicity, New York, 1957.

2. __________. Lectures on Ergodic Theory, New York, 1956.

3. Naimark, M.A. Normed Rings, Groningen, The Netherlands, 1964.

4. Parrott, S.K. Weighted Translation Operators, University of Michigan (Unpublished)

5. Pedersen, E.A. A Decomposition Theory for Rings of Operators, Louisiana State University (Unpublished)

6. Porcelli, P. Linear Spaces of Analytic Functions, Chicago, 1966.

7. Riesz, F., and Sz.-Nagy, B. Functional Analysis, New York, 1955.

8. Segal, I.E. Decompositions of Operator Algebras, I, Memoirs of the American Mathematical Society, No. 9, 1951

9. Stone, M.H. Linear Transformations in Hilbert Space and their Applications to Analysis, New York, 1966.

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Robert Michael Ristroph was born in Alexandria, Virginia on February 6, 1943. He attended Jesuit High School in New Orleans, Louisiana and became a National Merit Scholar. He did his undergraduate studies at Louisiana State University, where he received his Bachelor of Science in General Studies in August, 1963.

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